Nested algebraic Bethe ansaetze for orthogonal and symplectic spin chains

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Abstract

In this thesis the nested algebraic Bethe ansatz technique is applied to various orthogonal and symplectic closed and open spin chain models. Each spin chain considered is regarded as a representation of an underlying quantum group algebra, and expressions for eigenvectors of transfer matrices associated to these models are constructed using the algebra relations, reducing the problem to a set of Bethe equations. The specific models considered are the Ol'shanskii twisted Yangian spin chain, where $\mathfrak{gl}_n$ bulk symmetry is broken to orthogonal or symplectic symmetry; the MacKay twisted Yangian spin chain, an open spin chain with bulk orthogonal or symplectic symmetry and various boundary types; and the q-deformed orthogonal or symplectic closed spin chain. For the first and third cases, a closed 'trace formula' expression for the eigenvector is also provided.

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Declaration

This thesis contains material from [GMR19], which was written in collaboration with Prof. Niall MacKay and Dr Vidas Regelskis; and [GR20a] and [GR20b], which were written in collaboration with Dr Vidas Regelskis. Further, no claim of original work is made for Chapter 1, as well as Sections 2.1, 2.2.1, 3.1, 3.2.1, 3.2.2 and 4.1. Otherwise, I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Chapter 1

Introduction

Perfectly balanced between the simplicity of completely solvable models and the complexity of many body interactions, spin chains appear in many, often surprising, physical contexts—not just the original quantum statistical mechanics but also, as we shall see, the classical counterparts [On44] [Ba82], as well as quantum field theory [MZ03] and even certain Markov chains [Sa94]. The quest to understand these models gave birth to new algebras which underpin the theory and are today an active area of investigation. The algebraic Bethe ansatz and spin chains lie at the heart of this story of solvable physics that has come together over the last 90 years, from the dawn of quantum mechanics to the present day.

An appropriate starting point for this story is the very first spin chain model. Now often referred to as the Heisenberg model [He28], it is an attempt at modelling the interactions between electrons in a lattice of ions. Consider a lattice of static atoms, each with a single free electron, so that the resulting Schrödinger equation contains an interaction term for each pair of charged particles. Following earlier work of Heitler and London [HL27] on the H_2 bond, Heisenberg approached this problem perturbatively, considering the case in which all free electrons are paired to individual ions—or rather, the case in which atoms are infinitely separated from one another—to be the order zero approximation to the model. In this case the energy eigenfunctions of the electrons are identical to those of the hydrogen atom, each localised about a single ion. Interactions between electrons are introduced at first order but, from calculations of Heitler and London, the 'exchange integral'—the amplitude for the exchange of electrons at different sites—was found to decay exponentially with distance. With this, the resulting interaction Hamiltonian for this 'first order' model consists solely of interactions between electrons at *neighbouring* sites, with opposite spins. In other words, we arrive at a 'first order' Hamiltonian given essentially by

$$H = -\frac{J}{2} \sum_{<i,j>} (I - P_{ij}),$$

where the sum is taken over all neighbouring pairs of atoms, and P_{ij} is the operator that exchanges states at positions i and j.

Heisenberg in his paper laments that the model is in general not solvable and resorts to a statistical analysis of the energy levels. However, Bloch [Bl30] made some progress towards an exact solution. When considering the simplest possible lattice—a finite one-dimensional chain of L atoms with periodic boundary conditions—Bloch showed that the translational invariance of the model naturally led to wave-like eigenstates. These spin waves, or *magnons* in their particle interpretation, describe a single down spin in a sea of up spins. That is, for a chain of length L,

$$|k\rangle = \sum_{x=1}^{L} e^{ikx} |\uparrow\rangle_1 \cdots |\downarrow\rangle_x \cdots |\uparrow\rangle_L$$

describes the spin wave with wavenumber k, the allowed values of which are quantised by the periodic boundary conditions. The number of magnons is a symmetry of the Hamiltonian, and so these states provide a basis for constructing the eigenstates of the chain.

This idea was developed further by Bethe [Be31], who solved the model using his now-famous ansatz. Bethe began his analysis by calculating all possible two-magnon states, including the states with interacting spins, as well as an additional set of solutions with complex conjugate wavenumbers. With a two-magnon state of the form

$$|k_1, k_2\rangle = \sum_{x_1 < x_2} \left(A_1 e^{ik_1x_1 + ik_2x_2} + A_2 e^{ik_2x_1 + ik_1x_2}\right) |\uparrow\rangle_1 \cdots |\downarrow\rangle_{x_1} \cdots |\downarrow\rangle_{x_2} \cdots |\uparrow\rangle_L,$$

Bethe found that this would be an eigenstate only if the wavenumbers satisfied

$$\frac{A_1}{A_2} = \frac{z_1 - z_2 + i}{z_1 - z_2 - i}, \qquad \text{where} \quad z_j = \frac{1}{2}\cot\left(\frac{k_j}{2}\right).$$

Then, with the two magnon states fully mapped out, he gave an ansatz for the general r-magnon state which, crucially, had built-in the hypothesis that the phase difference between each r-magnon state depended only on phase differences between the two-magnon interacting states. Incredibly, he found that this method gave all possible eigenstates to arbitrary precision and for any number of atoms, provided the wave numbers satisfied a set of algebraic equations, now known as *Bethe equations*.

The theory of spin waves persisted, but significant developments in solution techniques for spin chains would not occur until much later, and would come from other, seemingly unrelated, areas of physics. We turn our attention to the Ising model—an earlier, classical version of Heisenberg's spin chain where 'spins' may only take one of two possible values (rather than a linear combination), and the simplest example in a broader theory of lattice models. The one-dimensional case was solved by Ising himself [Is25], and attention turned to more general lattice types in the 1940s. From this theory emerged the transfer matrix method, which allows one to write the partition function for the entire lattice in terms of a product of identical *transfer matrices*, each acting only on a single row or column. The full partition function can then be well-approximated in terms of the highest eigenvalue of this transfer matrix. Kramers and Wannier made use of this technique in their study of the 2D Ising model [KW41], which was built on by Onsager in [On44], who was able to find all transfer matrix eigenvectors.

In the 1950s and 60s the Bethe ansatz re-emerged, initially to solve the anisotropic Heisenberg spin chain (XXZ model) [Or58]. However, again, the significant developments in the theory came from a different area of physics, this time in the scattering of quantum particles with a delta-function interaction. Since the collisions between particles are completely elastic, there are only two possible outcomes for a single collision: the particles either retain their momenta, or they completely exchange momenta. Moreover, it was shown by McGuire [Mc64] that the full scattering matrix for N particles could be calculated exactly from these 2-particle interactions, given that the two possible factorisations of the 3-particle scattering were equal. Essentially, this particular model admitted 'factorised scattering' into the individual 2-particle interactions. This was built on by Yang [Ya67], who wrote the 2-particle S-matrix,

$$S_{ij} \propto I - \frac{icP_{ij}}{p_i - p_j},$$

where P_{ij} permutes the two states and p_i and p_j are the momenta of the two particles, and showed that it satisfied the relation

$$S_{jk}S_{ik}S_{ij} = S_{ij}S_{ik}S_{jk}. \tag{1.0.1}$$

Following Lieb and Liniger [LL63] before him, Yang showed that the spectral problem for this model was identical to that of a spin chain model and was thus able to apply the Bethe ansatz. Lieb also made use of the Bethe ansatz in studying the ice model [Li67], a simplified lattice model of the hydrogen bonding in ice. Connections were rapidly being found between these three theories, and this culminated in Baxter's solution of the eight-vertex model [Ba72]. In this paper Baxter introduced a transfer matrix which depended on a free parameter, and showed that, by tuning this parameter, previously studied solvable models including the XYZ (fully anisotropic) spin chain and the ice model appeared as specific cases. The model consisted of a directed graph on the square lattice, with each configuration of edges around a vertex assigned a certain energy level. This eight-vertex model allowed eight of the possible 16 energy levels to be nonzero. Summarising this data in a matrix R, Baxter was able to construct the transfer matrix as a partial trace of a product of these R matrices. He then showed that transfer matrices of different parameters would commute, and thus would be simultaneously diagonalisable, only if the R matrices satisfied a version of (1.0.1), the *Yang-Baxter equation*.

Before arriving at the quantum inverse scattering theory, we must first introduce its classical counterpart. The inverse scattering transform began as a novel solution to the Korteweg-de Vries equation, which governs the evolution of shallow water waves [GGKM67]. It was known [ZK65] that the equation yielded solitary wave, or *soliton*, solutions: travelling waves which are localised in space and do not dissipate. The method makes use of the already known inverse problem of reconstructing a potential from scattering data in quantum mechanics, due to [GL51], but in this case the 'potential' is the soliton solution itself. It was then Lax [La68] who showed how to generalise the method to other PDEs, and this was put into practice throughout the 70s as various solvable PDEs were put into this framework, including the nonlinear Schrödinger equation [SZ72] and the sine-Gordon field theory [AKNS73].

The quantisation of this theory was investigated throughout the 70s. It was discovered [Za77] that interaction between sine-Gordon solitons obeyed factorised scattering in the sense of Yang and McGuire which, in the process, revealed a new solution of the Yang-Baxter equation. The task of quantising the inverse scattering method itself was then undertaken by the Leningrad school, and from it came the quantum inverse scattering method [STF79] [TF79], or *algebraic Bethe ansatz*. Linking Baxter's theory to the theory of inverse scattering, the technique makes use of the Yang-Baxter equation not just to show the commutation of transfer matrices, but also in the construction of their joint eigenvectors. Indeed, it was shown that the eigenvector could be written in a form hardly more complicated than an expression for a harmonic oscillator eigenstate, with matrix elements of the monodromy matrix playing the role of the magnonic creation and annihilation operators.

The algebraic Bethe ansatz saw a lot of interest in the following decades, as it provided a framework for solving physical models that relied mainly on algebraic relations stemming from the Yang-Baxter equation, rather than particular physical properties of the models themselves. Indeed, any model for which an R-matrix—that is, a solution of the Yang-Baxter equation—had been found was a candidate for solution by the algebraic Bethe ansatz. As a result, the technique was quickly generalised to models with higher rank symmetry algebras compared to, for example, the $\mathfrak{su}_2$ symmetry of the XXX model. One of these developments was Kulish and Reshetikhin's *nested algebraic Bethe ansatz* (NABA) solution of the $\mathfrak{gl}_3$ [KR82] and, soon after, $\mathfrak{gl}_n$ spin chains [KR83]. It was found that the transfer matrix diagonalisation problem for $\mathfrak{gl}_n$ could be reduced to that for $\mathfrak{gl}_{n-1}$ and, inductively, down to the $\mathfrak{gl}_2$ problem, which is identical to the $\mathfrak{su}_2$ chain.

Through the study of the algebraic Bethe ansatz and factorised scattering, it became clear that solutions of the Yang-Baxter equation led directly to solvable physical models and, while many of these solutions had been uncovered via the physical theory, no formal mathematical classification had been achieved. The task of classifying these solutions was undertaken by Drinfel'd. First, in a paper with Belavin, Drinfel'd was able to classify non-degenerate solutions of the classical Yang-Baxter equation [BD82]. They showed that these corresponded to representations of simple Lie algebras, as well as belonging to one of three categories: rational, trigonometric or elliptic. Drinfel'd then showed that these classical solutions could be deformed to give solutions of the quantum Yang-Baxter equation. In doing this, Drinfel'd [D88] built up a new algebraic theory of 'quantum groups', in which the Lie algebra itself was deformed by this process. He named the rational quantum groups *Yangians*, while the trigonometric case, which was discovered independently by Jimbo [J85], became known as quantised enveloping algebras. The elliptic case had been studied by Sklyanin [Sk82] in the $\mathfrak{sl}_2$ case, and was extended to any simple Lie algebra by Felder [Fe94].

Spin chains based on these R-matrices were studied using the nested algebraic Bethe ansatz in the following works. For rational spin chains, the even orthogonal case was solved in [DVK87] for the vector representation. The symplectic case in the vector representation, as well as the odd orthogonal case in the spinor representation was solved in [Rs85]. A number of cases were studied in [MR97], including the orthosymplectic Lie superalgebra. Nevertheless, the $\mathfrak{gl}_n$ case is by far the most well-understood. In [TV94] (see also [TV13]), a non-recursive 'trace' formula was given for the eigenvector in terms of its Bethe roots. The $\mathfrak{gl}_n$ results were revisited in [BR08], where the super-rational and super-trigonometric cases in any representation were studied together, and a trace formula was given.

The investigation of boundary conditions would also be initiated during the 80s. Cherednik [Ch84] looked at the factorised scattering problem on the half-line, introducing a matrix to encode the boundary conditions of the model. He then went on to show that factorised scattering could be preserved so long as this matrix satisfied another Yang-Baxter-type equation with the R-matrix, now known as the boundary Yang-Baxter equation or *reflection equation*. Sklyanin [Sk88] applied this to the context of spin chains, and extended the idea to a system with two boundaries, a bounded chain of spins. He then went on to apply this theory to the XXZ spin chain, showing how an appropriate Hamiltonian with suitable boundary interaction terms could be extracted from this construction, and ultimately be diagonalised it using the algebraic Bethe ansatz.

Sklyanin's method gave a way of constructing commuting transfer matrices for these bounded systems starting from an R-matrix of the type used in the periodic chain and the reflection matrices which encoded the left and right boundary conditions, the K-matrix and dual K-matrix. Moreover, Sklyanin introduced the algebraic framework for understanding these models in the same paper, defining in the process boundary analogues of quantum groups. Following this, another type of algebra with similar properties to Sklyanin's, known as the *twisted Yangian*, was introduced by Ol'shanskii [Ol92]. From an mathematical perspective, the construction of these algebras mimicked a common construction of a Lie subalgebra from a Lie algebra using one of its involutions, which results in a *symmetric pair*. For the rational case, the K-matrices for the remaining symmetric pairs of simple Lie algebras were found in [MS01], and the corresponding twisted Yangian algebras were studied in a follow up work [M02].

Sklyanin's algebraic Bethe ansatz was extended to a nested algebraic Bethe ansatz for a $\mathfrak{gl}_n$ chain in [DVG94] and further to the supersymmetric cases in [BR09]. The orthogonal [GP16, Go18] and symplectic [GKR05] cases were studied. However, certain boundary conditions did not admit an algebraic Bethe ansatz solution due to the problem of defining a suitable vacuum vector from which to build the eigenstates. Use of gauge transformations allowed this problem to be mitigated somewhat [GM05], but in general an analytical Bethe ansatz-type approach [CYSW14] was necessary to find expressions for eigenvalues and Bethe equations for these systems.

In this thesis we solve some of the outstanding rational open spin chains using a nested algebraic Bethe ansatz method. We will start by reviewing the original algebraic Bethe ansatz for the rational Heisenberg spin chain before introducing the rational quantum groups which underpin the theory of closed chains, the Yangians associated with the Lie algebras $\mathfrak{gl}_n$, $\mathfrak{so}_{2n}$ and $\mathfrak{sp}_{2n}$, as well as their representation theory. We then describe the construction of the transfer matrix and Hamiltonian for open chains, and list the symmetric pairs for simple Lie algebras in preparation for Chapters 2 and 3.

In the remaining Chapters we systematically introduce quantum groups and their associated spin chains, and then proceed to solve them using the nested algebraic Bethe ansatz. In Chapter 2 we start with the *closed* $\mathfrak{gl}_n$ spin chain before repeating this process for the Ol'shanskii twisted Yangian spin chain, in the even cases. Similarly for Chapter 3 we focus first on the $\mathfrak{gl}_n$ reflection algebra and its spin chain before leading into the MacKay twisted Yangian spin chains, in the even cases. Finally, in Chapter 4 we give the NABA for a trigonometric (q-deformed) *closed* $U_q(\mathfrak{so}_{2n})$ or $U_q(\mathfrak{sp}_{2n})$ spin chain.

1.1 The algebraic Bethe ansatz

1.1.1 Construction of the transfer matrix

We begin by reviewing the algebraic Bethe ansatz for the original Heisenberg spin chain, in order to explain the general argument, set some of the notation and, moreover, to motivate the introduction of the underlying algebraic structures of spin chains. Despite the problems this causes for the completeness of the Bethe ansatz, we will choose to introduce the simplest possible spin chain, forgoing generalisations such as quasi-periodic boundary conditions, the higher spin case, or spectral parameter shifts, as these are unnecessary for understanding the overall method. This will largely follow Faddeev's lecture notes on the algebraic Bethe ansatz [Fa96].

The space of quantum states of the spin chain is an ℓ-fold tensor product of individual spin sites, spin-$\frac{1}{2}$ representations of $\mathfrak{su}_2$,

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 = (\mathbb{C}^2)^{\otimes \ell}.$$

Denote by e_i, for $i = 1, 2$ in this case, the basis vectors of each individual $\mathbb{C}^2$ space. We will denote by e_{ij} the elementary matrices, with the i, j entry being 1, and all other entries being zero.

We then define the *permutation operator* by the relation $P(a \otimes b) = (b \otimes a)$ for any $a, b \in \mathbb{C}^2$. With respect to our basis, this is given as a matrix by

$$P := \sum_{i,j=1}^{2} e_{ij} \otimes e_{ji} \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2).$$

We also introduce here the subscript notation to denote the tensor factors on which a matrix acts nontrivially. For example, $P_{23} = I \otimes P \in \mathrm{End}((\mathbb{C}^2)^{\otimes 3})$, and $P_{13} = \sum_{i,j} e_{ij} \otimes I \otimes e_{ji}$. In general numerical subscripts will denote spin chain sites, while we will also use subscripts a, a_i etc. to denote certain *auxiliary spaces*, which are crucial to the algebraic Bethe ansatz technique. From its defining property, the permutation operator satisfies $P^2 = I$, $PM_1P = M_2$ for any $M \in \mathrm{End}(\mathbb{C}^2)$ and $\mathrm{tr}_1\, P = I$, where tr_1 denotes the partial trace over the first space.

With this, we define Yang's R-matrix

$$R(u) = I - P\,u^{-1} \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)[u^{-1}],$$

which is invertible and satisfies the Yang-Baxter equation

$$R_{12}(u-v)R_{13}(u-w)R_{23}(v-w) = R_{23}(v-w)R_{13}(u-w)R_{12}(u-v). \tag{1.1.1}$$

The parameter u here is referred to as the *spectral parameter* and at this point we consider it to be indeterminate, but it may be thought of as a complex number throughout. The R-matrix also has various other important symmetry properties which will be discussed in later chapters. On the basis $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$, this R-matrix is given by

$$R(u) = \begin{pmatrix} 1-u^{-1} & 0 & 0 & 0 \\ 0 & 1 & -u^{-1} & 0 \\ 0 & -u^{-1} & 1 & 0 \\ 0 & 0 & 0 & 1-u^{-1} \end{pmatrix}. \tag{1.1.2}$$

We now introduce an *auxiliary space* $V_a \equiv \mathbb{C}^2$, and construct the *monodromy matrix* for the fundamental chain,

$$T_{a,1\ldots\ell}(v) = R_{a1}(v)R_{a2}(v)\cdots R_{a\ell}(v) \in \mathrm{End}(V_a \otimes \mathcal{H})[v^{-1}].$$

This is best understood from the perspective of factorised scattering theory: the monodromy matrix is the factorised S-matrix for the interaction between a test particle, represented by the auxiliary space V_a, and ℓ particles in a line, where the 2-particle S-matrix is given simply by the R-matrix. The test particle travels in a loop around the closed chain, arriving back at its original location, completing the full monodromy to give the above matrix. As a result of the Yang-Baxter equation (1.1.1), the monodromy matrix satisfies the *RTT relation*,

$$R_{a_1a_2}(u-v)T_{a_1}(u)T_{a_2}(v) = T_{a_2}(v)T_{a_1}(u)R_{a_1a_2}(u-v). \tag{1.1.3}$$

Note that above we have omitted the subscripts $1\ldots\ell$ from the monodromy matrix; we will adopt this convention throughout. The RTT relation is the cornerstone of the algebraic Bethe ansatz technique and we will return to it shortly.

We now define the *transfer matrix*, in the sense of Baxter, by taking the trace over the auxiliary space of the monodromy matrix to obtain an operator which acts on $\mathrm{End}(\mathcal{H})$ only:

$$t(v) := \mathrm{tr}_a\, T_a(v) \in \mathrm{End}(\mathcal{H})[v^{-1}].$$

Crucially, as a result of the RTT relation (1.1.3), transfer matrices of different parameter values mutually commute. This can be seen simply by multiplying from the left of (1.1.3) by $(R_{a_1a_2}(u-$

$v))^{-1}$, taking the partial trace over $V_{a_1} \otimes V_{a_2}$, and using its cyclicity. This implies that the coefficients of $t(v)$ all mutually commute, and can therefore be simultaneously diagonalised. For example, from the transfer matrix we can recover the Heisenberg spin chain Hamiltonian, arriving back at the spin chain interpretation of the model:

$$H = t^{(\ell-1)}(t^{(\ell)})^{-1} = \sum_{k=1}^{\ell-1} P_{k,k+1} + P_{\ell,1},$$

where here $t^{(j)}$ is the coefficient of v^{-j} in the polynomial expansion of $t(v)$.

1.1.2 Diagonalisation of the transfer matrix

The eigenvectors will be constructed from the action of monodromy matrix elements on a vacuum state, similar in spirit to the construction of the energy eigenstates of the harmonic oscillator. We decompose the monodromy matrix in the auxiliary space,

$$T(v) = \begin{pmatrix} a(v) & b(v) \\ c(v) & d(v) \end{pmatrix}.$$

Here we are viewing the operators $a(v), b(v), c(v), d(v)$ as operators which act on the spin chain only—that is, they act on a one-dimensional subspace of the auxiliary space, which may be ignored. In terms of these operators, the transfer matrix is $t(v) = a(v) + d(v)$.

The remaining elements $b(v)$ and $c(v)$ will be thought of as creation and annihilation operators respectively. However, we must first define a 'vacuum' state from which to construct our eigenstates. That is, a state in $\mathcal{H}$ that is annihilated by $c(v)$, while also a simultaneous eigenstate of $a(v)$ and $d(v)$. The RTT relation and the existence of the vacuum state may be thought of as the two necessary ingredients for applying the algebraic Bethe ansatz to closed spin chains. In this case, we may define such a state simply by $\eta := (e_1)^{\otimes \ell} \in \mathcal{H}$, the state in which all spins are aligned.

The properties of η are

$$c(v)\eta = 0 \qquad a(v)\eta = \lambda_1(v)\eta \qquad d(v)\eta = \lambda_2(v)\eta, \tag{1.1.4}$$

where $\lambda_1(v) = (1 - v^{-1})^{\ell}$, and $\lambda_2(v) = 1$. To see this, consider first the action of $T(v)$ on $e_1 \otimes \eta = (e_1)^{\otimes \ell+1}$. Since this state is completely symmetric, the permutation operator acts as the identity and the result is

$$T(v)(e_1 \otimes \eta) = (1 - v^{-1})^{\ell}(e_1 \otimes \eta),$$

giving the first two identities above.

The action on $e_2 \otimes \eta$ is slightly more complicated but, thinking again of the monodromy matrix as $T(v) = (I - P_{a1}v^{-1}) \cdots (I - P_{a\ell}v^{-1})$, there can be only one term in the resulting expression which leaves the e_2 in the auxiliary space, namely the one in which we take the identity matrix from each factor. Hence, η is also unchanged, and so we obtain the required eigenvalue for $d(v)$.

With this, we are ready to construct our transfer matrix eigenvectors. Pick $m \in \mathbb{N}$ and let $\boldsymbol{u} = (u_1, \dots, u_m) \in \mathbb{C}^m$ be a tuple of distinct nonzero parameters. We then define the *Bethe vector*

$$\Phi(\boldsymbol{u}) := b(u_1) \cdots b(u_m)\, \eta. \tag{1.1.5}$$

This constitutes the Bethe 'ansatz' for the transfer matrix eigenstates, and the goal is now to show how the transfer matrix acts on this vector. We will use the relations between the monodromy matrix elements to move $a(v)$ and $d(v)$ rightward through the creation operators. These relations are obtained from the RTT relation (1.1.3) and the relevant ones will be, for $u \neq v$,

$$a(v)\, b(u) = \left(1 - \frac{1}{u-v}\right) b(u)\, a(v) + \frac{1}{u-v} b(v)\, a(u) \tag{1.1.6}$$

$$d(v)\, b(u) = \left(1 + \frac{1}{u-v}\right) b(u)\, d(v) - \frac{1}{u-v} b(v)\, d(u) \tag{1.1.7}$$

$$b(v)\, b(u) = b(u)\, b(v). \tag{1.1.8}$$

The general method may be illustrated by the $m = 1$ case. Here we simply act with $a(v)$ and $d(v)$ individually on the Bethe vector $\Phi(u_1) = b(u_1)\eta$, commute these operators with the single creation operator and act diagonally on the vacuum state via (1.1.4). That is,

$$a(v)\, \Phi(u_1) = \lambda_1(v) \left(1 - \frac{1}{u_1 - v}\right) \Phi(u_1) + \frac{\lambda_1(u_1)}{u_1 - v} b(v)\eta$$

$$d(v)\, \Phi(u_1) = \lambda_2(v) \left(1 + \frac{1}{u_1 - v}\right) \Phi(u_1) - \frac{\lambda_2(u_1)}{u_1 - v} b(v)\eta.$$

We see that the right hand side contains two terms: one which is proportional to the Bethe vector with its parameter u_1 unchanged, and another term in which the parameters have been swapped. Since we require this process to diagonalise the transfer matrix for any value of the parameter v, these terms, or rather their sum, must vanish. The equation resulting from this condition is the *Bethe equation* for u_1, in this case given simply by $\lambda_1(u_1) = \lambda_2(u_1)$, which is a sufficient condition for $\Phi(u_1)$ to be an eigenvector of the transfer matrix.

The exchange relations (1.1.6-1.1.7) therefore each contain both a *wanted term*, in which the two operators retain their parameters, and a single *unwanted term*, in which the parameters are swapped. We will make use of this terminology in the $m > 1$ case below.

Indeed, we begin by extrapolating the exchange relations (1.1.6–1.1.8) to the case of multiple creation operators. Starting with expression $a(v)\, b(u_1) \cdots b(u_m)$, and repeatedly applying the exchange relation (1.1.6), we can move the operator $a(\cdot)$ through the creation operators until it is the rightmost operator. The terms of the resulting expression may then be grouped according to the spectral parameter of $a(\cdot)$. Indeed, the term containing $a(v)$ is given by

$$\prod_{i=1}^{m} \left(1 - \frac{1}{u_i - v}\right) b(u_1) \cdots b(u_m)\, a(v).$$

There can only be one such term, as we can allow no parameter swaps at each exchange of $a(v)$ and $b(u_i)$. That is, we take the 'wanted term' from each exchange.

All other terms contain $a(u_i)$ for some $1 \le i \le m$, and are all unwanted terms. In other words, we have, introducing notation $b(\boldsymbol{u}) := b(u_1)\cdots b(u_m)$,

$$a(v)\,b(\boldsymbol{u}) = \prod_{i=1}^{m}\left(1 - \frac{1}{u_i - v}\right) b(\boldsymbol{u})\,a(v) + \sum_{i=1}^{m} U_i(v;\boldsymbol{u})a(u_i), \tag{1.1.9}$$

where $U_i(v;\boldsymbol{u})$ belongs to the algebra spanned by the creation operators, noting that the exchange relation (1.1.8) ensures that this algebra is closed. This is important, as it ensures that the unwanted terms are unique up to reordering of the creation operators.

To calculate an expression for the unwanted terms, it is common to use the following argument. Consider first the result of applying the exchange relation a single time:

$$a(v)\,b(\boldsymbol{u}) = \left(\left(1 - \frac{1}{u_1 - v}\right) b(u_1)\,a(v) + \frac{1}{u_1 - v} b(v)a(u_1)\right) b(u_2)\cdots b(u_m).$$

The first term above will lead to the wanted term as before. However, by the same logic, the only contribution to $U_1(v;\boldsymbol{u})a(u_1)$ is obtained in the same way, by starting with the second term in the above expression and again taking the term from each subsequent exchange in which $a(u_1)$ retains its parameter. Therefore,

$$U_1(v;\boldsymbol{u}) = \frac{1}{u_1 - v}\prod_{i=2}^{m}\left(1 - \frac{1}{u_i - u_1}\right) b(v)b(u_2)\cdots b(u_m).$$

The remaining unwanted terms may be calculated by symmetry. Indeed, let $\sigma \in S_m$, and denote $\boldsymbol{u}_\sigma = (u_{\sigma(1)}, \dots, u_{\sigma(m)})$. The commutation of the creation operators (1.1.8) implies that $b(\boldsymbol{u}_\sigma) = b(\boldsymbol{u})$. Further, let σ_j denote the cyclic permutation defined by $\sigma_j(i) = j + i - 1 \mod \ell$. We may apply the previous logic to the creation operators with permuted parameters:

$$a(v)\,b(\boldsymbol{u}) = a(v)\,b(\boldsymbol{u}_{\sigma_j}) = \left(\left(1 - \frac{1}{u_j - v}\right) b(u_j)\,a(v) + \frac{1}{u_j - v} b(v)a(u_j)\right) b(u_{j+1})\cdots b(u_{j-1}),$$

resulting in an expression for $U_j(v;\boldsymbol{u})$:

$$U_j(v;\boldsymbol{u}) = \frac{1}{u_j - v}\prod_{i=2}^{m}\left(1 - \frac{1}{u_i - u_j}\right) b(v)b(u_{j+1})\cdots b(u_{j-1}).$$

Hence, we arrive at

$$a(v)\,b(\boldsymbol{u}) = \prod_{i=1}^{m}\left(1 - \frac{1}{u_i - v}\right) b(\boldsymbol{u})a(v) + \sum_{i=1}^{m} \frac{1}{u_i - v}\prod_{j\neq i}\left(1 - \frac{1}{u_j - u_i}\right) b(v)\,b(u_{\sigma_i(2)})\cdots b(u_{\sigma_i(m)})a(u_i). \tag{1.1.10}$$

An equivalent relation for $d(v)$ may be found in the same way, yielding

$$d(v)\,b(\boldsymbol{u}) = \prod_{i=1}^{m}\left(1+\frac{1}{u_i-v}\right)b(\boldsymbol{u})d(v) - \sum_{i=1}^{m}\frac{1}{u_i-v}\prod_{j\neq i}\left(1+\frac{1}{u_j-u_i}\right)b(v)\,b(u_{\sigma_i(2)})\cdots b(u_{\sigma_i(m)})d(u_i). \tag{1.1.11}$$

We now have enough information to give the full action of the transfer matrix on the Bethe vector $\Phi(\boldsymbol{u})$. Acting with (1.1.10) and (1.1.11) directly on the vacuum vector, we recall that the vacuum vector is an eigenvector of the diagonal elements of the monodromy matrix (1.1.4). Hence,

$$\begin{aligned}
a(v)\,\Phi(\boldsymbol{u}) &= \lambda_1(v)\prod_{i=1}^{m}\left(1-\frac{1}{u_i-v}\right)\Phi(\boldsymbol{u}) + \sum_{i=1}^{m}\frac{\lambda_1(u_i)}{u_i-v}\prod_{j\neq i}\left(1-\frac{1}{u_j-u_i}\right)\Phi((\boldsymbol{u}_\sigma)_{u_i\to v}),\\
d(v)\,\Phi(\boldsymbol{u}) &= \lambda_2(v)\prod_{i=1}^{m}\left(1+\frac{1}{u_i-v}\right)\Phi(\boldsymbol{u}) - \sum_{i=1}^{m}\frac{\lambda_2(u_i)}{u_i-v}\prod_{j\neq i}\left(1+\frac{1}{u_j-u_i}\right)\Phi((\boldsymbol{u}_\sigma)_{u_i\to v}),
\end{aligned}$$

where the subscript in $(\boldsymbol{u}_\sigma)_{u_i\to v}$ denotes the replacement of u_i with v. Summing these expressions gives

$$t(v)\,\Phi(\boldsymbol{u}) = \Lambda(v;\boldsymbol{u})\,\Phi(\boldsymbol{u}) + \sum_{i=1}^{m}\left[\frac{\lambda_1(u_i)}{u_i-v}\prod_{j\neq i}\left(1-\frac{1}{u_j-u_i}\right) - \frac{\lambda_2(u_i)}{u_i-v}\prod_{j\neq i}\left(1+\frac{1}{u_j-u_i}\right)\right]\Phi((\boldsymbol{u}_\sigma)_{u_i\to v}), \tag{1.1.12}$$

where

$$\Lambda(v;\boldsymbol{u}) = \lambda_1(v)\prod_{i=1}^{m}\left(1-\frac{1}{u_i-v}\right) + \lambda_2(v)\prod_{i=1}^{m}\left(1+\frac{1}{u_i-v}\right).$$

It is now clear that $\Phi(\boldsymbol{u})$ is an eigenvector of the transfer matrix, with eigenvalue $\Lambda(v;\boldsymbol{u})$, if the terms in square brackets in (1.1.12) vanish for each i. This condition results in the Bethe equations

$$\frac{\lambda_1(u_i)}{\lambda_2(u_i)} = \prod_{j\neq i}\frac{u_j-u_i+1}{u_j-u_i-1} \qquad \text{for} \quad 1\leq i\leq m,$$

or, inserting the expressions for $\lambda_1(u_i)$ and $\lambda_2(u_i)$,

$$\left(1-\frac{1}{u_i}\right)^{\ell} = \prod_{j\neq i}\frac{u_j-u_i+1}{u_j-u_i-1} \qquad \text{for} \quad 1\leq i\leq m.$$

1.2 Yang-Baxter algebras

As we saw in the previous section, almost all steps of the algebraic Bethe ansatz, as well as the commutation of the transfer matrices, stemmed from the RTT relation, itself a generalisation of the Yang-Baxter equation. This naturally leads to the idea of a *Yang-Baxter algebra*, an algebra generated by T-matrices, with the RTT relation as its defining property. Let $R(u,v)$ be an invertible solution of the Yang-Baxter equation

$$R_{ab}(u,v)R_{ac}(u,w)R_{bc}(v,w) = R_{bc}(v,w)R_{ac}(u,w)R_{ab}(u,v).$$

Then we regard the Yang-Baxter algebra to be the associative unital algebra generated by $T(u)$, with relations

$$R_{ab}(u,v)T_a(u)T_b(v) = T_b(v)T_a(u)R_{ab}(u,v)$$

and no others. Specifically, we will regard the Yang-Baxter algebra $\mathcal{A}$ to be generated by $t_{ij}^{(r)}$, where $r \in \mathbb{Z}_{\geq 0}$ and $i,j = 1,\dots n$, which form the coefficients of formal power series in u^{-1}, which in turn are considered to be the matrix elements of $T(u)$. That is,

$$t_{ij}(u) := \sum_{r\geq 0} t_{ij}^{(r)} u^{-r} \in \mathcal{A}[[u^{-1}]],$$

and the generating matrix is defined by

$$T(u) = \sum_{i,j=1}^{n} e_{ij} \otimes t_{ij}(u) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathcal{A}[[u^{-1}]].$$

Any representation of this algebra then could be said to define a monodromy matrix, the trace of which $\operatorname{tr} T(u)$ defines commuting transfer matrices. Any R matrix may be used to define a Yang-Baxter algebra, and the map $T_a(u) \mapsto R_{ab}(u,v)$ even provides a representation of the algebra, due to the Yang-Baxter equation. Therefore any solution of the Yang-Baxter equation defines commuting transfer matrices, and a classification of these solutions provides a list of potentially solvable models.

As an example of a Yang-Baxter algebra, we introduce the Yangian $Y(\mathfrak{gl}_n)$, adhering closely to [Mo07].

1.2.1 The Yangian $Y(\mathfrak{gl}_n)$

We begin by briefly reviewing the definition and representation theory of $U(\mathfrak{gl}_n)$, the universal enveloping algebra of $\mathfrak{gl}_n$.

Definition 1.2.1. *The universal enveloping algebra of $\mathfrak{gl}_n$, $U(\mathfrak{gl}_n)$, is the unital associative $\mathbb{C}$-algebra generated by elements 1 and E_{ij} for $1 \leq i,j \leq n$, satisfying relations*

$$E_{ij}E_{kl} - E_{kl}E_{ij} = \delta_{kj}E_{il} - \delta_{il}E_{kj}. \tag{1.2.1}$$

For any Lie algebra, representations of $U(\mathfrak{g})$ may be thought of as representations of $\mathfrak{g}$ and vice versa; we will tend to refer to them in the latter manner.

First, we introduce finite dimensional highest weight representations of $\mathfrak{gl}_n$. A $\mathfrak{gl}_n$-module V is a *highest weight module* if there exists a vector $\eta \in V$ such that V is generated by the action of $\mathfrak{gl}_n$

on η, and

$$\begin{aligned} E_{ii}\eta =& \lambda_i \eta, \quad \text{for} \quad 1 \le i \le n \qquad \text{and} \\ E_{ij}\eta =& \quad 0, \quad \text{for} \quad 1 \le i < j \le n. \end{aligned}$$

Then η is the *highest weight vector* and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ is the *highest weight*. Highest weight representations may be defined abstractly as quotients of the Verma module for any $\lambda \in \mathbb{C}^n$, but will be finite dimensional if and only if $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\ge 0}$ for each $1 \le i \le n-1$. This reflects the condition that these are dominant integral weights. These finite dimensional modules are irreducible and we denote the $\mathfrak{gl}_n$-module with highest weight $\lambda \in \mathbb{C}^n$ by $L(\lambda)$.

There is another way of defining $U(\mathfrak{gl}_n)$, which will help introduce some of the concepts which will be valuable when studying the Yangian.

We first construct a matrix of generators

$$E := \sum_{i,j=1}^{n} e_{ij} \otimes E_{ij} \in \mathrm{End}(\mathbb{C}^n) \otimes U(\mathfrak{gl}_n).$$

This may be viewed as an $n \times n$ matrix of $U(\mathfrak{gl}_n)$ elements through the isomorphism $\mathrm{End}(\mathbb{C}^n) \otimes U(\mathfrak{gl}_n) \xrightarrow{\sim} \mathrm{Mat}_{n\times n}(U(\mathfrak{gl}_n))$ as $\mathbb{C}$-algebras. The defining relations for $U(\mathfrak{gl}_n)$ may then be written in terms of this E. First, recall the permutation matrix $P \in \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n)$, which satisfies $P(a \otimes b) = b \otimes a$ for all $a, b \in \mathbb{C}^n$, given explicitly by

$$P := \sum_{i,j=1}^{n} e_{ij} \otimes e_{ji}.$$

Then the defining relation of $U(\mathfrak{gl}_n)$ is equivalent to

$$E_1 E_2 - E_2 E_1 = E_1 P - P E_1 \in \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n) \otimes U(\mathfrak{gl}_n). \tag{1.2.2}$$

The subscripts here denote the tensor factor in which the matrix acts nontrivially.

We now proceed to introduce the Yangian $Y(\mathfrak{gl}_n)$ in its RTT presentation.

Definition 1.2.2. *The $\mathfrak{gl}_n$-Yangian, $Y(\mathfrak{gl}_n)$ is a unital associative $\mathbb{C}$-algebra generated by 1 and $t_{ij}^{(r)}$ for $1 \le i, j \le n$ and $r \in \mathbb{N}$, satisfying relations*

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)}, \tag{1.2.3}$$

for each $1 \le i, j, k, l \le n$, $r, s \in \mathbb{Z}_{\ge 0}$ with the convention that $t_{ij}^{(0)} = \delta_{ij} 1$.

For each $1 \le i, j \le n$, define the formal power series $t_{ij}(u) := \sum_{r=0}^{\infty} t_{ij}^{(r)} u^{-r} \in Y(\mathfrak{gl}_n)[[u^{-1}]]$. Then the defining relation of the Yangian (1.2.3) is equivalent to

$$(u - v)[t_{ij}(u), t_{kl}(v)] = t_{kj}(u) t_{il}(v) - t_{kj}(v) t_{il}(u). \tag{1.2.4}$$

This form of the defining relations can be thought of as analogous to the relations (1.1.6–1.1.8) which we used in the algebraic Bethe ansatz. However, we will also apply the same treatment to these formal series generators of $Y(\mathfrak{gl}_n)$ that we did earlier for the generators of $U(\mathfrak{gl}_n)$. First, recall Yang's R-matrix

$$R(u) := I - u^{-1}P \in \mathrm{End}(\mathbb{C}^n)[u^{-1}].$$

Define now the *generating matrix* for $Y(\mathfrak{gl}_n)$

$$T(u) := \sum_{i,j=1}^{n} e_{ij} \otimes t_{ij}(u) \in \mathrm{End}(\mathbb{C}^n) \otimes Y(\mathfrak{gl}_n)[[u^{-1}]].$$

Then the relation (1.2.4) is equivalent to the *RTT relation*,

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v) \tag{1.2.5}$$

which is now a relation in $\mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n) \otimes Y(\mathfrak{gl}_n)[[u^{-1}, v^{-1}]]$.

The Yangian generating matrix is therefore an abstraction of the monodromy matrix for spin chains, and the spin chain defines a particular representation of the Yangian. Much of the argument of the algebraic Bethe ansatz, as well as the commutation of the transfer matrices, depends only on the RTT relation. The exception to this is the existence of the vacuum vector (1.1.4) which depends on the spin chain itself, or, in other words, the representation of $Y(\mathfrak{gl}_n)$.

Before defining these representations, it will be useful to first introduce some automorphisms and anti-automorphisms of the Yangian. Working from Molev [Mo07], we have the following result.

Lemma 1.2.3. *The following define automorphisms of* $Y(\mathfrak{gl}_n)$*:*

$$\begin{aligned}
\tau_c &: T(u) \mapsto T(u-c) && \textit{for} \quad c \in \mathbb{C} \\
\mu_f &: T(u) \mapsto f(u)T(u) && \textit{for} \quad f(u) = 1 + \cdots \in \mathbb{C}[[u^{-1}]] \\
g_A &: T(u) \mapsto A\,T(u)\,A^{-1} && \textit{for} \quad A \in GL(n)
\end{aligned}$$

Proof. For τ_c, the elements $t_{ij}(u-c)$ may be constructed as a formal series in $(u-c)^{-1}$ with the same coefficients as $t_{ij}(u)$. Then, the fact that these shifted elements satisfy relations (1.2.4) is a consequence of the relations depending only on the difference $(u-v)$. However, it remains to show that τ_c actually defines a map between elements of the Yangian. For this we make use the formal series identity $\frac{1}{1-c/u} = (1 + c/u + (c/u)^2 + \ldots)$, which allows us to view $t_{ij}(u-c)$ as a series in u^{-1} with zeroth coefficient equal to δ_{ij}.

For μ_f, multiplying the RTT relation on both sides by $f(u)f(v)$ yields the necessary result.

For g_A we use the fact that the R-matrix commutes with $A \otimes A$, which amounts to the fact that $P(A \otimes A) = (A \otimes A)P$. Conjugating the RTT relation by $(A \otimes A)$ then yields the required result. □

Lemma 1.2.4. *The following define anti-automorphisms of* $Y(\mathfrak{gl}_n)$*:*

$$sign : T(u) \mapsto T(-u), \tag{1.2.6}$$

$$tran : T(u) \mapsto (T(u))^{\intercal}, \tag{1.2.7}$$

$$S : T(u) \mapsto T^{-1}(u). \tag{1.2.8}$$

Proof. We need to show that these satisfy the relations of $Y(\mathfrak{gl}_n)$ with the multiplication in the Yangian algebra reversed.

$$R_{12}(u-v)\widetilde{T}_2(v)\widetilde{T}_1(u) = \widetilde{T}_1(u)\widetilde{T}_2(v)R_{12}(u-v) \tag{1.2.9}$$

The first we obtain by conjugating by the permutation operator, swapping spaces 1 and 2. Note however that we have $R_{12}(u) = R_{21}(u)$ from the definition of the permutation operator. Then

$$R_{12}(u-v)T_2(u)T_1(v) = T_1(v)T_2(u)R_{12}(u-v).$$

Setting $u = -v'$ and $v = -u'$, we obtain (1.2.9).

For the transpose, applying the transpose to both spaces in the RTT relation, we have

$$T_1(u)^{\intercal_1}T_2(v)^{\intercal_2}R_{12}(u-v)^{\intercal_1\intercal_2} = R_{12}(u-v)^{\intercal_1\intercal_2}T_2(v)^{\intercal_2}T_1(u)^{\intercal_1}.$$

Then, as the permutation operator and therefore R-matrix is symmetric, we see that this is identical to (1.2.9).

The existence of (left and right) inverses of $T(u)$ is a consequence of the fact that the zeroth coefficient of $T(u)$, as a formal series in u^{-1}, is the identity in $\mathrm{End}(\mathbb{C}^n) \otimes Y(\mathfrak{gl}_n)$. Indeed, by expanding the relations $T^L(u)T(u) = I \otimes 1$ and $T(u)T^R(u) = I \otimes 1$ as formal series in u^{-1} we may build the left and right inverses $T^L(u)$ and $T^R(u)$ inductively. For example, at level-0 we have $(T^L)^{(0)} = I \otimes 1$, and at level-1, $(T^L)^{(0)}T^{(1)} + (T^L)^{(1)}T^{(0)} = 0$, giving $(T^L)^{(1)} = -T^{(1)}$. Then, the left and right inverses can be shown to be identical by $T^L(u)T(u)T^R(u) = T^L(u) = T^R(u) =: T^{-1}(u)$.

To show that this defines an anti-automorphism, from the RTT relation (1.2.5) we need only multiply from the left by $T_1^{-1}(u)T_2^{-1}(v)$ to arrive at (1.2.9). □

Remark 1.2.5. The inverse $T^{-1}(u)$ may also be constructed in manner analogous to the cofactor construction for matrices. Indeed, we introduce the *quantum determinant* $\mathrm{qdet}\, T(u)$ of the matrix $T(u)$ as (see Definition 1.6.5 and Proposition 1.6.6 in [Mo07])

$$\mathrm{qdet}\, T(u) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\, t_{1\sigma(1)}(u-n+1)\cdots t_{n\sigma(n)}(u).$$

It is a formal power series in u^{-1} with coefficients in $Y(\mathfrak{gl}_n)$. In fact, it can be shown ([Mo07] Theorem 1.7.5) that these coefficients generate the centre of $Y(\mathfrak{gl}_n)$. Going further, one may define $Y(\mathfrak{sl}_n)$ as the quotient of $Y(\mathfrak{gl}_n)$ by the relation $\mathrm{qdet}\, T(u) = 1$, which then has trivial centre.

We then construct the *quantum comatrix* (adjugate matrix) elementwise by taking the quantum determinant of the generating matrix with one row and column missing. Specifically,

$$\widehat{t}_{ij}(u) = (-1)^{i+j}\mathrm{qdet}\,|T|_{ji}(u),$$

where $|T|_{ji}(u)$ denotes the $(n-1)\times(n-1)$ matrix obtained by removing the j^{th} row and i^{th} column from the matrix $T(u)$. It then follows by [Mo07] Proposition 1.9.2 that

$$\widehat{T}(u)T(u-n+1) = \mathrm{qdet}\,T(u).$$

The inverse matrix $T^{-1}(u)$ with matrix elements $t'_{ij}(u)$ is then given by

$$t'_{ij}(u) = (\mathrm{qdet}\,T(u+n-1))^{-1}\cdot\widehat{t}_{ij}(u+n-1).$$

Consider now the relation (1.2.3) with $r=0$ and $s=1$. We find

$$[t_{ij}^{(1)}, t_{kl}^{(1)}] = \delta_{kj}t_{il}^{(1)} - \delta_{il}t_{kj}^{(1)}.$$

Comparing with the defining relations of $U(\mathfrak{gl}_n)$ (1.2.1) we see that the level 1 generators, with the identity, form $U(\mathfrak{gl}_n)$ as a subalgebra within $Y(\mathfrak{gl}_n)$. That is, the map $E_{ij}\mapsto t_{ij}^{(1)}$ defines an embedding $U(\mathfrak{gl}_n)\hookrightarrow Y(\mathfrak{gl}_n)$. We can also go in the other direction, from the Yangian to $U(\mathfrak{gl}_n)$. This map is called the *evaluation homomorphism*, and is defined by $t_{ij}(u)\mapsto\delta_{ij}+u^{-1}E_{ij}$ The evaluation homomorphism allows us to extend any representation of $U(\mathfrak{gl}_n)$ (that is, any representation of $\mathfrak{gl}_n$) to a representation of the Yangian. Please note that throughout this work, we will instead use the homomorphism

$$ev : t_{ij}(u)\mapsto\delta_{ij}-u^{-1}E_{ji}, \tag{1.2.10}$$

which is a composition of the above map with the anti-automorphisms (1.2.6) and (1.2.7). This is sometimes referred to as the *twisted evaluation homomorphism*. We will also make use of the homomorphism $ev_c = ev\circ\tau_c$.

While this does define an action of the Yangian, one might wonder what we have gained by throwing out all the higher levels of the Yangian algebra. The key is what lies at the heart of all quantum groups: the coproduct.

We first introduce these ideas in the context of $U(\mathfrak{gl}_n)$. In the context of associative algebras, the *coproduct* is a homomorphism from the algebra to the tensor product of the algebra with itself. In $U(\mathfrak{gl}_n)$ the standard choice is $\Delta : U(\mathfrak{gl}_n)\to U(\mathfrak{gl}_n)\otimes U(\mathfrak{gl}_n)$, $\Delta(E_{ij}) = E_{ij}\otimes 1 + 1\otimes E_{ij}$, and $\Delta(1) = 1\otimes 1$. This is recognisable as the "addition of spin" rule in quantum mechanics.

For the Yangian, we make the following choice of coproduct:

$$\Delta(t_{ij}(u)) = \sum_{k=1}^{n} t_{ik}(u)\otimes t_{kj}(u). \tag{1.2.11}$$

This defines a map on the formal series; the action on the original generators can be found by expanding the series and using the coproduct's linearity. In particular, we have

$$\Delta(t_{ij}^{(1)}) = t_{ij}^{(1)} \otimes 1 + 1 \otimes t_{ij}^{(1)},$$

showing that the Yangian coproduct is a deformation of the standard $U(\mathfrak{gl}_n)$ coproduct.

One crucial property of the Yangian coproduct is its lack of cocommutativity. That is, swapping the ordering of the tensor factors gives a different coproduct

$$\Delta^{opp}(t_{ij}(u)) = \sum_{k=1}^{n} t_{kj}(u) \otimes t_{ik}(u).$$

This is in contrast to $U(\mathfrak{gl}_n)$, and is the property which makes the Yangian a 'quantum group'.

If we define also the *counit* $\epsilon(t_{ij}(u)) = \delta_{ij}$ and the *antipode* $S(T(u)) = T^{-1}(u)$, the Yangian becomes a Hopf algebra. These will not see active use in the construction of spin chains, but do show that the tensor product defined by the coproduct satisfies standard properties that we might expect.

With this coproduct we see that, while the action of the Yangian through the evaluation homomorphism is not different than the action of $U(\mathfrak{gl}_n)$ on a single module, the action of the Yangian on a tensor product of modules is more interesting, and this will allow us to make use of the Yangian's unique properties in the study of spin chains.

We now proceed to introduce the representation theory of the Yangian. Crucially we are interested in representations which contain a vector with properties (1.1.4).

Definition 1.2.6. *A representation V of $Y(\mathfrak{gl}_n)$ is called a lowest weight representation if there exists a nonzero vector $\eta \in V$ such that $V = Y(\mathfrak{gl}_n)\eta$ and*

$$\begin{aligned} t_{ij}(u)\eta &= 0 \qquad &&\text{for} \quad 1 \le j < i \le n \quad \text{and} \\ t_{ii}(u)\eta &= \lambda_i(u)\eta &&\text{for} \quad 1 \le i \le n, \end{aligned}$$

where $\lambda_i(u)$ is a formal power series in u^{-1} with a constant term equal to 1. The vector η is called the lowest vector of V, and the n-tuple $\lambda(u) = (\lambda_1(u), \ldots, \lambda_n(u))$ is called the lowest weight of V.

The distinction of lowest weight here, rather than highest weight, is due to the choice of which generators annihilate the lowest weight vector. Choosing elements below the diagonal for this gives the lowest weight module definition, whereas a module in which elements above the diagonal play this role would be a highest weight module. Due to our earlier choice of evaluation homomorphism which includes a transpose, these lowest weight modules are connected to highest weight modules of $U(\mathfrak{gl}_n)$.

Indeed, we now proceed to construct the spin chain from $U(\mathfrak{gl}_n)$ modules, and use the evaluation homomorphism to define a Yangian action on it. Recall $L(\lambda)$, the finite dimensional irreducible highest weight $U(\mathfrak{gl}_n)$-module with weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ and highest weight vector η. Applying the evaluation homomorphism (1.2.10), $L(\lambda)$ is a Yangian lowest weight module with lowest weight vector η and lowest weight $\lambda(u) = (\lambda_1(u), \ldots, \lambda_n(u))$ where

$$\lambda_i(u) = 1 - \lambda_i u^{-1}. \tag{1.2.12}$$

We will refer to this as an *evaluation module*.

We may then build the spin chain out of these modules, with Yangian action defined via the coproduct and evaluation homomorphism. We also include the shift automorphism to allow for more generality, and this will be relevant for the nested Bethe ansatz in Chapter 2. Fix $\ell \in \mathbb{N}$ and consider the tensor product of finite dimensional irreducible $\mathfrak{gl}_n$ modules

$$L(\lambda^{(1)}) \otimes L(\lambda^{(2)}) \otimes \ldots \otimes L(\lambda^{(\ell)}), \tag{1.2.13}$$

denoting their highest weight vectors by $\eta^{(k)}$ for $1 \leq k \leq \ell$. We define a Yangian action on this space as follows. First, define recursively $\Delta^{(k)} = (id \otimes \cdots \otimes id \otimes \Delta) \circ \Delta^{(k-1)}$ with $\Delta^{(2)} := \Delta$. Then, the action is defined by

$$t_{ij}(u) \mapsto (ev_{c_1} \otimes \cdots \otimes ev_{c_\ell}) \circ \Delta^{(\ell)}(t_{ij}(u))$$

where $c_k \in \mathbb{C}$ for $1 \leq k \leq \ell$. This is a Yangian lowest weight module with lowest weight vector $\eta := \eta^{(1)} \otimes \cdots \otimes \eta^{(\ell)}$, and lowest weight $\lambda(u) = (\lambda_1(u), \ldots, \lambda_n(u))$ given by

$$\lambda_i(u) = \prod_{k=1}^{\ell} \left(1 - \frac{\lambda_i^{(k)}}{u - c_k}\right).$$

As a Yangian module, we will denote this by

$$L := L(\lambda^{(1)})_{c_1} \otimes L(\lambda^{(2)})_{c_2} \otimes \ldots \otimes L(\lambda^{(\ell)})_{c_\ell}, \tag{1.2.14}$$

including the parameter shifts in the notation. The *binary property* of the tensor products of Yangian modules states that, for a suitable choice of weights $\lambda_i^{(k)}$ and parameters c_k, the $Y(\mathfrak{gl}_n)$-module L is irreducible, see Theorem 6.5.8 in [Mo07].

Defining the *Lax operators* as

$$\mathcal{L}(u - c) := (id \otimes ev_c)(T(u)) = \sum_{i,j=1}^{n} e_{ij} \otimes \left(\delta_{ij} - \frac{E_{ji}}{u - c}\right), \tag{1.2.15}$$

the generating matrix $T(u)$ acts on the space L by

$$T_a(u) \cdot L = \left(\prod_{i=1}^{\ell} \mathcal{L}_{ai}(u - c_i)\right) L \in \mathrm{End}(\mathbb{C}^n) \otimes L[[u^{-1}]]. \tag{1.2.16}$$

In particular, in the case of $n = 2$ with vector representations at each site, that is, $\lambda = (1, 0)$, we arrive at the Heisenberg spin chain.

Note that in the product of non-commuting operators in (1.2.16), and in what follows, we use the convention that the operators are ordered left to right—that is, the leftmost operator above is $\mathcal{L}_{a1}(u - c_1)$. In order to denote the reversed product we will make use of a decreasing index, for example $\prod_{i=\ell}^{1}$.

Finally, it will be necessary to define the action of $T^{-1}(u)$ on lowest weight modules. Indeed, it follows (see the proof of Theorem 4.2 in [MR02]) from the definitions of $\operatorname{qdet} T(u)$, $\widehat{T}(u)$ and η that

$$t'_{ij}(u)\eta = 0 \quad \text{for} \quad 1 \le j < i \le n \quad \text{and} \quad t'_{ii}(u)\eta = \lambda'_i(u)\eta \quad \text{for} \quad 1 \le i \le n$$

with the "inverse-weights" $\lambda'_i(u)$ defined by

$$\lambda'_i(u) = \frac{\lambda_1(u+1)\cdots\lambda_{i-1}(u+i-1)}{\lambda_1(u)\cdots\lambda_i(u+i-1)}. \tag{1.2.17}$$

1.2.2 Orthogonal and symplectic Yangians

In this section we give equivalent results for the even orthogonal and symplectic Yangians. We will largely follow [AMR06], which also contains results for the odd orthogonal case. We again begin by introducing the universal enveloping algebras $U(\mathfrak{so}_{2n})$ and $U(\mathfrak{sp}_{2n})$. In fact, we may study both at once by introducing $U(\mathfrak{g}_{2n})$, which is equal to $U(\mathfrak{so}_{2n})$ or $U(\mathfrak{sp}_{2n})$ depending on a choice of sign. In what follows, we will use $\pm$ or $\mp$ to denote this choice of sign; the upper sign being the orthogonal case and the lower sign being the symplectic case. Additionally, let

$$\theta_i = \begin{cases} \pm 1 & \text{for} \quad 1 \le i \le n, \\ 1 & \text{for} \quad i > n, \end{cases}$$

and let $\theta_{ij} = \theta_i \theta_j$.

Definition 1.2.7. *The universal enveloping algebra of $\mathfrak{g}_{2n}$, $U(\mathfrak{g}_{2n})$, is the unital associative $\mathbb{C}$-algebra generated by elements 1, F_{ij} for $1 \le i, j \le 2n$, satisfying relations*

$$[F_{ij}, F_{kl}] = \delta_{jk}F_{il} - \delta_{il}F_{kj} + \theta_{ij}(\delta_{j\bar{l}}F_{k\bar{\imath}} - \delta_{i\bar{k}}F_{\bar{\jmath}l}), \tag{1.2.18}$$

$$F_{ij} + \theta_{ij}F_{\bar{\jmath}\bar{\imath}} = 0, \tag{1.2.19}$$

with $\bar{\imath} = 2n - i + 1$ and $\bar{\jmath} = 2n - j + 1$.

We may regard this as a subalgebra of $U(\mathfrak{gl}_{2n})$ by setting $F_{ij} = E_{ij} - \theta_{ij}E_{\bar{\jmath}\bar{\imath}}$. Going further, we define a particular transpose t by

$$e_{ij}^t = \theta_{ij}e_{\bar{\jmath}\bar{\imath}}. \tag{1.2.20}$$

This is identical to a regular matrix transpose followed by conjugation by the matrix $J = \sum_i \theta_i e_{i\bar{\imath}}$. As this is an anti-automorphism, following it by multiplication by -1 gives an (involutive) automorphism of $U(\mathfrak{gl}_{2n})$. Then, we may regard the $U(\mathfrak{g}_{2n})$ subalgebra as generated by elements of $U(\mathfrak{gl}_{2n})$ that are symmetric with respect to this automorphism. That is, $F = E - E^t$, where $F := \sum_{ij} e_{ij} \otimes F_{ij}$.

The matrix form of the defining relations is then

$$F_1F_2 - F_2F_1 = F_1(P - Q) - (P - Q)F_1 \qquad \text{and} \qquad F + F^t = 0, \tag{1.2.21}$$

where

$$Q := P^{t_1} = P^{t_2} = \sum_{i,j=1}^{2n} \theta_{ij} e_{ij} \otimes e_{\bar{\imath}\bar{\jmath}} \in \mathrm{End}(\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}). \tag{1.2.22}$$

The matrix Q has properties which we will frequently make use of in later chapters. First,

$$PQ = QP = \pm Q \qquad \text{and} \qquad Q^2 = 2nQ,$$

implying that Q is a projector. Additionally, recall that $PM_1 = M_2P$ for any $M \in \mathrm{End}(\mathbb{C}^{2n})$. Taking the transpose of this, we obtain a pair of relations for Q:

$$Q\,M_1 = Q\,M_2^t, \qquad M_1\,Q = M_2^t Q. \tag{1.2.23}$$

For any n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ there exists an irreducible highest weight representation $V(\lambda)$ of the Lie algebra $\mathfrak{g}_{2n}$. In particular, $V(\lambda)$ is generated by a non-zero vector η such that

$$\begin{aligned} F_{ij}\,\eta &= 0 && \text{for} \quad 1 \le i < j \le 2n \quad \text{and} \\ F_{ii}\,\eta &= \lambda_i\,\eta && \text{for} \quad 1 \le i \le n. \end{aligned}$$

The representation $V(\lambda)$ is finite-dimensional if and only if

$$\begin{aligned} \lambda_i - \lambda_{i+1} &\in \mathbb{Z}_+ && \text{for} \quad i = 1, \ldots, n-1 \quad \text{and} \\ \lambda_{n-1} + \lambda_n &\in \mathbb{Z}_+ && \text{if} \quad \mathfrak{g}_{2n} = \mathfrak{so}_{2n}, \\ \lambda_n &\in \mathbb{Z}_+ && \text{if} \quad \mathfrak{g}_{2n} = \mathfrak{sp}_{2n}. \end{aligned}$$

We now introduce the extended Yangian $X(\mathfrak{g}_{2n})$ and its representation theory, adhering closely to [AMR06]. First we introduce the *Zamolodchikov R-Matrix* [ZZ78],

$$R(u) = I - \frac{1}{u}P - \frac{1}{\kappa - u}Q, \quad \text{where} \quad \kappa = n \mp 1, \tag{1.2.24}$$

acting on $\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$. Note that the distinction between the orthogonal and symplectic cases is contained within the operator Q and the parameter κ, which is the dual Coxeter number.

The R-matrix is of course a solution of the Yang-Baxter equation,

$$R_{12}(u - v)\,R_{13}(u - z)\,R_{23}(v - z) = R_{23}(v - z)\,R_{13}(u - z)\,R_{12}(u - v), \tag{1.2.25}$$

but also satisfies $R(u)^t = R(\kappa - u)$, where $R(u)^t := R(u)^{t_1} = R(u)^{t_2}$, and

$$R(u)R^t(u+\kappa) = R^t(u+\kappa)R(u) = (1-u^{-2})I. \tag{1.2.26}$$

Following this, we may define the Yang-Baxter algebra associated to this R-matrix, which is known as the *extended Yangian* $X(\mathfrak{g}_{2n})$. Introduce elements $t_{ij}^{(r)}$ with $1 \le i,j \le 2n$ and $r \ge 0$ such that $t_{ij}^{(0)} = \delta_{ij}$. Combining these into formal power series $t_{ij}(u) = \sum_{r\ge 0} t_{ij}^{(r)} u^{-r}$, we can then form the generating matrix $T(u) = \sum_{i,j=1}^{2n} e_{ij} \otimes t_{ij}(u)$.

Definition 1.2.8. *The extended Yangian $X(\mathfrak{g}_{2n}$ is the unital associative $\mathbb{C}$-algebra generated by elements $t_{ij}^{(r)}$ with $1 \le i,j \le 2n$ and $r \in \mathbb{Z}_{\ge 0}$ satisfying the relation*

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v). \tag{1.2.27}$$

The Hopf algebra structure of $X(\mathfrak{g}_{2n})$ is given by

$$\Delta : t_{ij}(u) \mapsto \sum_{k=1}^{2n} t_{ik}(u) \otimes t_{kj}(u), \qquad S : T(u) \mapsto T^{-1}(u), \qquad \varepsilon : T(u) \mapsto I. \tag{1.2.28}$$

We now collect several useful facts about the algebra $X(\mathfrak{g}_{2n})$. The matrix $T(u)$ satisfies the matrix form of the cross-unitarity relation (1.2.26),

$$T(u)T^t(u+\kappa) = T^t(u+\kappa)T(u) = z(u)I, \tag{1.2.29}$$

where $z(u)$ is a formal series in u^{-1} with coefficients that can be shown to be central in $X(\mathfrak{g}_{2n})$ [AMR06]. In this sense, this relation plays a similar role to the quantum determinant for $Y(\mathfrak{gl}_n)$. Going further, one may define the $\mathfrak{g}_{2n}$ *Yangian* $Y(\mathfrak{g}_{2n})$ by taking the quotient of $X(\mathfrak{g}_{2n})$ by the relations $T(u)T^t(u+\kappa) = T^t(u+\kappa)T(u) = I$, setting $z(u) = 1$, and it can be shown that this resulting algebra has trivial centre.

Let $G(2n)$ denote the Lie group associated to $\mathfrak{g}_{2n}$, and let $c \in \mathbb{C}$ and $f(u) \in \mathbb{C}[[u^{-1}]]$. The following are also automorphisms of $X(\mathfrak{g}_{2n})$,

$$\tau_c : T(u) \mapsto T(u-c), \quad \text{for} \quad c \in \mathbb{C} \tag{1.2.30}$$

$$\mu_f : T(u) \mapsto f(u)T(u) \quad \text{for} \quad f(u) = 1 + \cdots \in \mathbb{C}[[u^{-1}]] \tag{1.2.31}$$

$$g_B : T(u) \mapsto BT(u)B^t \quad \text{for} \quad B \in GL(2n) \quad \text{with} \quad BB^t = 1. \tag{1.2.32}$$

The proofs are identical to those for Lemma 1.2.4. The proof for g_B requires that $B \otimes B$ commutes with $R(u)$, in particular with Q. Indeed, $B_1B_2Q = B_1B_1^tQ = Q = QB_1B_2$.

The following define anti-automorphisms of $X(\mathfrak{g}_{2n})$:

$$T(u) \mapsto T(-u), \qquad T(u) \mapsto T^t(u), \qquad T(u) \mapsto (T(u))^{-1}. \tag{1.2.33}$$

Note that in this case, the inverse anti-automorphism can be built from the transpose due to the cross-unitarity relation (1.2.29).

We may define lowest weight representations for $X(\mathfrak{g}_{2n})$ analogously to those for $Y(\mathfrak{gl}_n)$.

Definition 1.2.9. *A representation V of $X(\mathfrak{g}_{2n})$ is called a lowest weight representation if there exists a non-zero vector $\eta \in V$ such that $V = X(\mathfrak{g}_{2n})\eta$ and*

$$t_{ij}(u)\eta = 0 \quad \textit{for} \quad 1 \leq j < i \leq 2n \quad \textit{and} \quad t_{ii}(u)\eta = \lambda_i(u)\eta \quad \textit{for} \quad 1 \leq i \leq 2n, \tag{1.2.34}$$

where $\lambda_i(u)$ is a formal power series in u^{-1} with a constant term equal to 1. The vector η is called the lowest vector of V and the $2n$-tuple $\lambda(u) = (\lambda_1(u), \ldots, \lambda_{2n}(u))$ is called the lowest weight of V.

The Yangian $X(\mathfrak{g}_{2n})$ contains the universal enveloping algebra $U(\mathfrak{g}_{2n})$ as a Hopf subalgebra. An embedding $U(\mathfrak{g}_{2n}) \hookrightarrow X(\mathfrak{g}_{2n})$ is given by

$$F_{ij} \mapsto \tfrac{1}{2}(t_{ij}^{(1)} - \theta_{ij} t_{\bar{j}\bar{i}}^{(1)}) \tag{1.2.35}$$

for all $1 \leq i, j \leq 2n$. However, in contrast to $Y(\mathfrak{gl}_n)$, there is no surjective homomorphism from $X(\mathfrak{g}_{2n})$ onto the algebra $U(\mathfrak{g}_{2n})$, that is, there is no evaluation homomorphism [D85]. As a consequence, not every irreducible finite-dimensional representation of $\mathfrak{g}_{2n}$ can be extended to a representation of $X(\mathfrak{g}_{2n})$. In Chapter 3 we tackle the problem of defining suitable representations with which to build a spin chain using the R-matrix fusion procedure.

This concludes this chapter's discussion of RTT algebras, however, this will be resumed in Chapter 4, where we will introduce the quantum loop algebras $U_q(\mathfrak{Lgl}_n)$ and $U_q(\mathfrak{Lg}_{2n})$. These algebras are the trigonometric equivalents to the rational Yangian algebras introduced above, and share a similar representation theory.

1.3 Open spin chains

In this section we give a brief introduction to open spin chains. We construct the open Heisenberg spin chain with reflective boundary conditions using a transfer matrix, and then proceed to describe the challenges associated with the algebraic Bethe ansatz for an open spin chain. The underlying algebras of these spin chains will be introduced in their respective chapters, but we conclude with a table of symmetric pairs for simple Lie algebras over $\mathbb{C}$, which will correspond to the different types of reflective boundary conditions we will be interested in.

1.3.1 The open Heisenberg spin chain

The techniques discussed so far are relevant for closed chains, those with periodic and quasi-periodic boundary conditions. However, we may also introduce reflective boundary conditions to the theory.

Consider once again the quantum state space

$$\mathcal{H} = (\mathbb{C}^2)^{\otimes \ell}.$$

Our open spin chain will make use of the same state space. However, while the bulk of the spin chain will have the same nearest neighbour interaction as the closed Heisenberg spin chain, the interaction Hamiltonian will include terms which describe the interaction of the leftmost and rightmost sites with fixed boundaries. This Hamiltonian will be obtained from a transfer matrix, which will again be the partial trace of a monodromy matrix, the construction of which now follows.

Working once again from the perspective of scattering theory, we will construct the monodromy matrix by sending a test particle through the spin chain. The particle interacts with each spin site before reflecting off the right boundary, returning back through the spin chain with a reversed momentum. It then reflects off the left boundary to return to its original position, completing the monodromy of the chain.

We therefore introduce matrices $K(u), \widetilde{K}(u) \in \mathrm{End}(\mathbb{C}^2)[u^{-1}]$ to represent the interaction of the test particle with the right and left boundaries respectively, and the full monodromy of the chain is given by

$$S_{a,1\dots\ell}(v) = R_{a1}(v)\dots R_{a\ell}(v)K_a(v)R_{a\ell}(v)\dots R_{a1}(v)\widetilde{K}_a(v),$$

where once again $R(u) = 1 - Pu^{-1}$. This construction was first introduced by Sklyanin in [Sk88]. In his paper, Sklyanin showed that the partial trace of a monodromy matrix constructed in this way $\tau_{1\dots\ell}(v) = \mathrm{tr}_a\, S_{a,1\dots\ell}(v)$ gave a transfer matrix which enjoyed properties analogous to those of a transfer matrix for a periodic chain, which we outline below.

First, a nearest neighbour Hamiltonian may be extracted from this transfer matrix. Assuming that the constant part of $K(u)$ is equal to the identity matrix, the Hamiltonian can be found by taking the coefficient of $u^{-2\ell-1}$ of the transfer matrix. Indeed,

$$\tau^{(2\ell+1)} = -2\,\mathrm{tr}\,\widetilde{K}^{(0)}\left[\sum_{k=1}^{\ell-1} P_{i,i+1} - \tfrac{1}{2}K_\ell^{(1)} + \frac{\widetilde{K}_1^{(0)}}{\mathrm{tr}\,\widetilde{K}^{(0)}}\right] + \mathrm{tr}\,K^{(1)}.$$

Then, stripping the constant terms and scalar multiples, we arrive at a Hamiltonian given by the terms in square brackets above. This describes nearest neighbour interaction in the ‘bulk’ of chain, but also includes two terms which describe interactions with the left and right boundaries.

Second, these transfer matrices will commute, provided the matrices $K(u)$ and $\widetilde{K}(u)$ satisfy the *reflection equation*

$$R_{a_1a_2}(u-v)K_{a_1}(u)R_{a_1a_2}(u+v)K_{a_2}(v) = K_{a_2}(v)R_{a_1a_2}(u+v)K_{a_1}(u)R_{a_1a_2}(u-v),$$

and the *dual reflection equation*

$$R_{a_1a_2}(-u+v)\widetilde{K}^t_{a_1}(u)R_{a_1a_2}(-u-v+2)\widetilde{K}^t_{a_2}(v) = \widetilde{K}^t_{a_2}(v)R_{a_1a_2}(-u-v+2)\widetilde{K}^t_{a_1}(u)R_{a_1a_2}(-u+v)$$

respectively. These equations are analogous to the Yang-Baxter equation for periodic chains.

The algebraic Bethe ansatz may also be applied to this chain, in a way that is detailed in Sklyanin's original work, where he considered the XXZ version of the chain. Recall that in the periodic case, the starting point for the algebraic Bethe ansatz was the RTT relation (1.1.3) that was satisfied by the monodromy matrix, and this gave the relations between its matrix elements which were used to construct the ansatz.

In this case, however, we focus not on the full monodromy matrix, but instead on the matrix given by

$$\widetilde{S}(v) := R_{a1}(v) \dots R_{a\ell}(v) K_a(v) R_{a\ell}(v) \dots R_{a1}(v),$$

the monodromy matrix without the left boundary. The reason for this is a property of the reflection equation: solutions of the reflection equation may extended to give further solutions using representations of Yang-Baxter algebras, that is, solutions of the RTT relations. Indeed, for any $T(u)$ that satisfies the RTT relation (1.1.3) and is invertible, the matrix given by

$$T(u)K(u)T^{-1}(-u) \tag{1.3.1}$$

can be shown to satisfy the reflection equation. This is related to the co-ideal property of reflection algebras, which will be discussed in later chapters. For now we see that the monodromy matrix is an example of this, as $R^{-1}(-u) \propto R(u)$.

We then write

$$\widetilde{S}(u) = \begin{pmatrix} a(u) & b(u) \\ c(u) & d(u) \end{pmatrix},$$

and extract relations between the matrix elements $a(u), b(u), c(u)$ and $d(u)$. The eigenstate is built from $b(u)$ operators, in a similar way to the periodic case. The details of the calculation will not be given here, but there are two remaining crucial factors that will be relevant in future chapters.

First is the dual K-matrix. The dual K-matrix may be reinstated as a modification to the relations between matrix elements of $\widetilde{S}(v)$. For the Heisenberg chain the algebraic Bethe ansatz is still possible, however, this freedom does not extend to higher rank symmetry algebras such as $\mathfrak{gl}_n$. Indeed, it was shown in [BR09] that the $n \times n$ dual K-matrix could have not more degrees of freedom than the 2×2 one used in the Heisenberg chain. For this reason, in later chapters we will work only with the case $\widetilde{K}(u) = I$, the trivial solution of the dual reflection equation. As such, we drop the notation $\widetilde{S}$, as $S(u) = \widetilde{S}(u)$.

The second matter is that of the vacuum vector. For this system, the vacuum vector is unchanged from the closed chain, equal to $(e_1)^{\otimes \ell}$. However, the existence of the vacuum vector is not guaranteed: it depends on the extent to which the symmetry of the spin chain is broken by the boundary conditions. In particular, off-diagonal terms within the K-matrix cause a mixing of creation and annihilation operators, preventing the existence of an appropriate vacuum [GM05]. For this reason, we consider only diagonal K-matrices in later chapters.

Despite these restrictions, we are left with a considerable number of possible boundary types.

We will be interested in those that may be described by *symmetric pairs*, which describe the bulk Lie symmetry of the chain—that is, the choice of R-matrix—along with the resulting Lie symmetry after symmetry breaking by the boundary. We therefore conclude this section with a brief introduction to symmetric pairs.

1.3.2 Symmetric pairs

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, and let θ be an involutive automorphism of $\mathfrak{g}$. In fact, we will be interested in only the classical Lie algebras—the infinite families of simple Lie algebras. Discussion of the exceptional cases can be found in [He01]. As θ is involutive it must have eigenvalues of ± 1, and hence $\mathfrak{g}$ may be decomposed into the positive and negative eigenspaces of θ:

$$\mathfrak{g} = \mathfrak{g}^{\theta} \oplus \mathfrak{f}.$$

Here $\mathfrak{g}^{\theta}$ is the fixed point subalgebra of θ in $\mathfrak{g}$, that is, the $+1$ eigenspace. We then denote by $(\mathfrak{g}, \mathfrak{g}^{\theta})$ the symmetric pair of the Lie algebra and its fixed point subalegebra.

These involutions will almost all be *inner automorphisms*: conjugation by an element of the Lie group associated to the Lie algebra. Any *outer* automorphisms can be characterised as automorphisms of the Lie algebra's Dynkin diagram and, for classical Lie algebras, exist only in the cases of $\mathfrak{sl}_n$ (reflective symmetry of the entire diagram) and $\mathfrak{so}_{2n}$ (exchange of branched nodes), see [He01] pp. 514 Table II. A list of symmetric pairs and the associated involutive automorphisms is given in Table 1.3.2, adapted from [GR16].

In subsequent Chapters we will study rational quantum groups associated with these symmetric pairs. In Chapter 2 we study the Ol'shanskii twisted Yangian [Ol92], associated with symmetric pairs of type AI and AII. In Chapter 3 we study both the Molev-Ragoucy (extended) reflection algebra [MR02], associated to type AIII, and the MacKay twisted Yangians [M02] associated to types CI, CII, DI(a) and DIII.

Name	$\mathfrak{g}$	θ	$\mathfrak{g}^\theta$
AI (a)	$\mathfrak{sl}_{2n}$	$E \mapsto -E^{t+}$	$\mathfrak{so}_{2n}$
AI (b)	$\mathfrak{sl}_{2n+1}$	$E \mapsto -E^{t+}$	$\mathfrak{so}_{2n+1}$
AII	$\mathfrak{sl}_{2n}$	$E \mapsto -E^{t-}$	$\mathfrak{sp}_{2n}$
AIII	$\mathfrak{sl}_n$	$E \mapsto GEG^{-1}, G = \left(\begin{array}{c\|c} I_p & \\ \hline & -I_{n-p} \end{array}\right)$	$\mathfrak{sl}_p \oplus \mathfrak{sl}_{n-p} \oplus \mathbb{C}$
BI	$\mathfrak{so}_{2n+1}$	$F \mapsto GFG^{-1}, G = \left(\begin{array}{c\|c\|c} I_p & & \\ \hline & -I_{2n-2p+1} & \\ \hline & & I_p \end{array}\right)$	$\mathfrak{so}_{2p} \oplus \mathfrak{so}_{2n-2p+1}$
CI	$\mathfrak{sp}_{2n}$	$F \mapsto GFG^{-1}, G = \left(\begin{array}{c\|c} I_n & \\ \hline & -I_n \end{array}\right)$	$\mathfrak{gl}_n$
CII	$\mathfrak{sp}_{2n}$	$F \mapsto GFG^{-1}, G = \left(\begin{array}{c\|c\|c} I_p & & \\ \hline & -I_{2n-2p} & \\ \hline & & I_p \end{array}\right)$	$\mathfrak{sp}_{2p} \oplus \mathfrak{sp}_{2n-2p}$
DI (a)	$\mathfrak{so}_{2n}$	$F \mapsto GFG^{-1}, G = \left(\begin{array}{c\|c\|c} I_p & & \\ \hline & -I_{2n-2p} & \\ \hline & & I_p \end{array}\right)$	$\mathfrak{so}_{2p} \oplus \mathfrak{so}_{2n-2p}$
DI (b)	$\mathfrak{so}_{2n}$	$F \mapsto GFG^{-1}, G = \left(\begin{array}{c\|c\|c} & & J_{2p+1} \\ \hline & I_{2n-4p-2} & \\ \hline J_{2p+1} & & \end{array}\right)$	$\mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2n-2p-1}$
DIII	$\mathfrak{so}_{2n}$	$F \mapsto GFG^{-1}, G = \left(\begin{array}{c\|c} I_n & \\ \hline & -I_n \end{array}\right)$	$\mathfrak{gl}_n$

Table 1.1: Table of classical symmetric pairs, adapted from [He01] and [GR16], where I_r is the $r \times r$ identity matrix and J_r is the matrix with ones on the anti-diagonal and zeros elsewhere.

Chapter 2

Nested algebraic Bethe ansatz for the even twisted Yangian spin chain

In this chapter we first review the nested algebraic Bethe ansatz for a closed $\mathfrak{gl}_n$ spin chain. We then recall the definition of Ol'shanskii's twisted Yangian $Y^{\pm}(\mathfrak{gl}_{2n})$ as well as its representation theory, and use this to define an open spin chain and commuting transfer matrices which act on the spin chain. We use the nested algebraic Bethe ansatz to construct eigenvectors for this transfer matrix, finding the eigenvalues and Bethe equations. Finally, we give a 'trace formula' closed form expression for the constructed eigenvectors in terms of the Bethe roots.

2.1 Nested algebraic Bethe ansatz for a closed $Y(\mathfrak{gl}_n)$ spin chain

Recall $Y(\mathfrak{gl}_n)$ in its RTT presentation as defined in Definition 1.2.2, as well as the $Y(\mathfrak{gl}_n)$ module L defined by (1.2.14),

$$L := L(\lambda^{(1)})_{c_1} \otimes L(\lambda^{(2)})_{c_2} \otimes \ldots \otimes L(\lambda^{(\ell)})_{c_\ell}, \tag{2.1.1}$$

which is a lowest weight $Y(\mathfrak{gl}_n)$ module with lowest weight

$$\lambda_i(u) = \prod_{k=1}^{\ell} \left(1 - \frac{\lambda_i^{(k)}}{u - c_k}\right). \tag{2.1.2}$$

We will assume that the weights $\lambda^{(k)}$ and shifts c_k are such that this forms an irreducible representation of $Y(\mathfrak{gl}_n)$. Recall also that the RTT relation implies that $t(u) := \operatorname{tr} T(u)$ satisfies

$$[t(u), t(v)] = 0.$$

The goal of the nested algebraic Bethe ansatz is to construct eigenvectors of $t(u)$ in L. The approach follows that of Kulish and Reshetikin's original paper from 1983 [KR83], and more recently that of Belliard and Ragoucy [BR08] in which generalisations are given.

If we recall the algebraic Bethe ansatz from Chapter 1, we used a product of creation operators

$b(u)$ to construct an ansatz for the transfer matrix eigenstates. In doing so, we appeal to a version of the Poincaré-Birkhoff-Witt theorem, see Theorem 1.4.1 of [Mo07] for its proof.

Theorem 2.1.1. *Given an arbitrary linear order on the set of generators $t_{ij}^{(r)}$, any element of $Y(\mathfrak{gl}_n)$ can be uniquely written as a linear combination of ordered monomials in these generators.*

Combining this result with the properties of the lowest weight modules, for which the spin chain is an example, we see that any element of such a module may be constructed by acting on the lowest vector by the above-diagonal generators of the Yangian—those which sit above the diagonal in the generating matrix. Put more precisely, we have the following result.

Corollary 2.1.2. *Let L be a finite dimensional lowest weight $Y(\mathfrak{gl}_n)$-module with lowest weight vector η. Then, as a vector space, L is spanned by vectors of the form*

$$t_{i_1 j_1}^{(r_1)} \cdots t_{i_m j_m}^{(r_m)} \eta,$$

with $m \in \mathbb{N}$, and $r_k \in \mathbb{N}$ and $1 \leq i_k < j_k \leq n$ for each k.

To see this, we simply pick an ordering for the generators which place the 'below-diagonal' generators—the $t_{ij}^{(r)}$ with $i > j$—as the rightmost operators, followed by the diagonal generators, and finally the above-diagonal generators as the leftmost operators. Since the lowest weight vector is by definition an eigenvector of all below-diagonal and diagonal generators, those generators play no part in generating the remaining vectors and thus we need only consider the above-diagonal elements.

Of course, this result alone does not prove that the eigenvectors of the transfer matrix can be written in the form used in the algebraic Bethe ansatz. However, it justifies the use of generators above the diagonal in the generating matrix—that is, creation operators—in constructing eigenvectors without the need of involving the diagonal or below-diagonal elements.

In the case of $Y(\mathfrak{gl}_2)$ this led to a unique creation operator, with which we constructed eigenvectors of the transfer matrix. Moving to $Y(\mathfrak{gl}_n)$, however, there are $\frac{1}{2}n(n-1)$ creation operators, and the approach is no longer so simple: which ordering and linear combinations of creation operators will facilitate the steps of the algebraic Bethe ansatz, that is, the rightward movement of diagonal generators which make up the transfer matrix?

Kulish and Reshetikhin's nested algebraic Bethe ansatz [KR83] gives an answer to this question which makes use of a convenient property of the Yangian: namely that any $p \times p$ diagonal submatrix of $Y(\mathfrak{gl}_n)$ defines a $Y(\mathfrak{gl}_p)$ subalgebra within $Y(\mathfrak{gl}_n)$. It can then be shown that, by taking the creation operators one row at a time, the $Y(\mathfrak{gl}_n)$ problem can be reduced to another transfer matrix diagonalisation problem for $Y(\mathfrak{gl}_{n-1})$. Using the chain of subalgebras $Y(\mathfrak{gl}_n) \supset Y(\mathfrak{gl}_{n-1}) \supset \cdots \supset Y(\mathfrak{gl}_2)$, we arrive at the $Y(\mathfrak{gl}_2)$ case, to which a regular ABA may be applied.

2.1.1 Exchange relations

Recall the Yang R-matrix $R(u) = I - u^{-1}P \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)[[u^{-1}]]$, and the Yangian generating matrix $T_a(u) \in \mathrm{End}(V_a) \otimes Y(\mathfrak{gl}_n)[[u^{-1}]]$, where $V_a = \mathbb{C}^n$ is an auxiliary space. We will refer to $T_a(v)$ as the monodromy matrix, although strictly this name should be reserved for the representative of the $T_a(v)$ on the spin chain module.

As with the $\mathfrak{gl}_2$ case we begin by splitting the monodromy matrix into four operators. Crucially, however, in the $\mathfrak{gl}_n$ case these operators will be *matrices* in the auxiliary space rather than the scalars $a(v), b(v), c(v), d(v)$. We denote this 'reduced' auxiliary space by $V_a' = \mathbb{C}^{n-1}$ and further reductions by $V_a^{(k)} = \mathbb{C}^{n-k}$ for any $0 \leq k < n$, so that $V_a = V_a^{(0)}$ and $V_a' = V_a^{(1)}$. Accordingly, the monodromy matrix $T_a(u)$ splits into block matrices as follows:

$$T_a(u) = \left(\begin{array}{c|c} a(u) & B_a(u) \\ \hline C_a(u) & D_a(u) \end{array} \right), \tag{2.1.3}$$

where $a(u) = t_{11}(u)$ and

$$\begin{aligned} B_a(u) &= (t_{12}(u), \ldots, t_{1n}(u)) && \in (V_a')^* \otimes Y(\mathfrak{gl}_n)[[u^{-1}]], \\ C_a(u) &= (t_{21}(u), \ldots, t_{n1}(u))^T && \in V_a' \otimes Y(\mathfrak{gl}_n)[[u^{-1}]], \\ D_a(u) &= \begin{pmatrix} t_{22}(u) & \ldots & t_{2n}(u) \\ \vdots & \ddots & \vdots \\ t_{n2}(u) & \ldots & t_{nn}(u) \end{pmatrix} && \in \mathrm{End}(V_a') \otimes Y(\mathfrak{gl}_n)[[u^{-1}]]. \end{aligned}$$

In particular, $B_a(u)$ is a row-vector and $C_a(u)$ is a column-vector. It will be convenient to denote the matrix entries of $B_a(u)$ by $b_i(u)$ with $1 \leq i \leq n-1$, and similarly for $C_a(u)$ and $D_a(u)$. Additionally, we introduce a reduced R-matrix $R'(u)$ acting on $\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1}$,

$$R'(u) := I - u^{-1} \sum_{i,j=1}^{n-1} e'_{ij} \otimes e'_{ji} = I - u^{-1}P'.$$

The defining relations of $Y(\mathfrak{gl}_n)$ imply the following exchange relations for $a(v)$, $B_a(v)$ and $D_a(v)$:

$$a(v)B_{a_1}(u) = \frac{v-u+1}{v-u} B_{a_1}(u)\, a(v) - \frac{1}{v-u} B_{a_1}(v)\, a(u), \tag{2.1.4}$$

$$D_a(v)B_{a_1}(u) = B_{a_1}(u)\, D_a(v)\, R'_{aa_1}(v-u) + \frac{1}{v-u} B_{a_1}(v)\, D_a(u)\, P'_{aa_1}, \tag{2.1.5}$$

$$B_{a_1}(v)B_{a_2}(u) = \frac{v-u}{v-u-1} B_{a_2}(u) B_{a_1}(v)\, R'_{a_1a_2}(v-u), \tag{2.1.6}$$

along with an RTT relation

$$R'_{a_1a_2}(u-v)\, D_{a_1}(u)\, D_{a_2}(v) = D_{a_2}(v)\, D_{a_1}(u)\, R'_{a_1a_2}(u-v). \tag{2.1.7}$$

In particular, the coefficients of the matrix entries of $D_a(v)$ generate a subalgebra $Y(\mathfrak{gl}_{n-1}) \subset Y(\mathfrak{gl}_n)$ (note, however, that this is not a Hopf subalgebra). Two additional relations will be used, which can be stated more clearly in terms of individual matrix entries of $T_a(u)$. For any $1 \le i, j, k \le n-1$,

$$c_k(u)\, d_{ij}(v) = d_{ij}(v)\, c_k(u) - \frac{1}{u-v}\big(d_{kj}(u)\, c_i(v) - d_{kj}(v)\, c_i(u)\big), \tag{2.1.8}$$

$$[a(v), d_{ij}(u)] = \frac{1}{v-u}\big(b_j(u)\, c_i(v) - b_j(v)\, c_i(u)\big). \tag{2.1.9}$$

2.1.2 Exchange relations for multiple excitations

The exact construction of the Bethe vector, that is, the ansatz for the eigenvector of the transfer matrix will be given in a later section. However, largely it will involve a product of creation operators just as in the $\mathfrak{gl}_2$ case and, anticipating this, we first establish the exchange relations between the diagonal operators and the multiple product of creation operators which will appear in the ansatz.

Choose $m \in \mathbb{N}$ and introduce an m-tuple $\boldsymbol{u} = (u_1, \ldots, u_m)$ of formal parameters. Let the spaces $V'_{a_1}, \ldots, V'_{a_m}$ be copies of $V'_a \equiv \mathbb{C}^{n-1}$. The *creation operator* for m excitations is

$$B_{a_1 \ldots a_m}(\boldsymbol{u}) := B_{a_1}(u_1) \cdots B_{a_m}(u_m).$$

This is a row-vector, which lives in the tensor product of dual-spaces $(V'_{a_1})^* \otimes \ldots \otimes (V'_{a_m})^*$ with entries in $Y(\mathfrak{gl}_n)[[u_1^{-1}, \ldots, u_m^{-1}]]$. Note that we will also use $\boldsymbol{a}$ as a shorthand for action on the auxiliary spaces $a_1, \ldots, a_m$.

The parameters carried by $B_{\boldsymbol{a}}(\boldsymbol{u})$ may be exchanged by the braided R-matrix defined by

$$\check{R}'(u) := \frac{u}{u-1}\, R'(u) P'. \tag{2.1.10}$$

Indeed, from (2.1.6) we have

$$B_{a_1}(u_1) B_{a_2}(u_2) = B_{a_1}(u_2) B_{a_2}(u_1) \check{R}'_{a_1 a_2}(u_1 - u_2).$$

Consequently, for m excitations, we have that

$$B_{\boldsymbol{a}}(\boldsymbol{u}) = B_{\boldsymbol{a}}(\boldsymbol{u}_{i \leftrightarrow i+1})\, \check{R}'_{a_i a_{i+1}}(u_i - u_{i+1}) \quad \text{for} \quad 1 \le i \le m-1, \tag{2.1.11}$$

where $u_{i \leftrightarrow i+1}$ is the m-tuple $(u_1, \ldots, u_{i+1}, u_i, \ldots u_m)$. Going further, as any permutation $\sigma \in S_m$ is a sequence of transpositions, we may realise any permutation of the parameters $\boldsymbol{u}$ on the creation operator by a product of $\check{R}'$ matrices:

$$B_{\boldsymbol{a}}(\boldsymbol{u}) = B_{\boldsymbol{a}}(\boldsymbol{u}_\sigma) \check{R}'_{\boldsymbol{a}}[\sigma](\boldsymbol{u}), \tag{2.1.12}$$

where $\boldsymbol{u}_\sigma = (u_{\sigma(1)}, \ldots, u_{\sigma(m)})$ and $\check{R}'_{\boldsymbol{a}}[\sigma](\boldsymbol{u})$ is the product of $\check{R}'$ matrices required to enact this permutation. In what follows we will make use of the cyclic permutations $\sigma_j \in S_m$ with $\sigma_j(i) = i + j - 1 \mod m$. Another consequence of this relation is that the coefficients of $b_i(u)$ for each i combined form a closed subalgebra $\mathcal{B}$ of $Y(\mathfrak{gl}_n)$.

We are now ready to establish the exchange relations for the $a(v)$ and $D_a(v)$ operators, which generalise (2.1.4) and (2.1.5) to m excitations.

Lemma 2.1.3. *The following identities hold*

$$a(v)\, B_{\boldsymbol{a}}(\boldsymbol{u}) = \left(\prod_{i=1}^{m} \frac{v-u_i+1}{v-u_i}\right) B_{\boldsymbol{a}}(\boldsymbol{u})\, a(v) \\ - \sum_{j=1}^{m} \frac{1}{v-u_j}\, B_{\boldsymbol{a}}(\boldsymbol{u}_{\sigma_j, u_j\to v}) \operatorname*{Res}_{w\to u_j} \left[\left(\prod_{i=1}^{m} \frac{w-u_i+1}{w-u_i}\right) a(w)\right] \check{R}'_{\boldsymbol{a}}[\sigma_j](\boldsymbol{u}), \tag{2.1.13}$$

$$\operatorname{tr}_a D_a(v)\, B_{\boldsymbol{a}}(\boldsymbol{u}) = B_{\boldsymbol{a}}(\boldsymbol{u}) \operatorname{tr}_a T'_{a;\boldsymbol{a}}(v;\boldsymbol{u}) \\ - \sum_{j=1}^{m} \frac{1}{v-u_j}\, B_{\boldsymbol{a}}(\boldsymbol{u}_{\sigma_j, u_j\to v}) \operatorname*{Res}_{w\to u_j} \operatorname{tr}_a T'_{a;\boldsymbol{a}}(w;\boldsymbol{u}_{\sigma_j})\, \check{R}'_{\boldsymbol{a}}[\sigma_j](\boldsymbol{u}), \tag{2.1.14}$$

where

$$T'_{a;\boldsymbol{a}}(v;\boldsymbol{u}) := D_a(v)\, R'_{aa_m}(v-u_m)\cdots R'_{aa_1}(v-u_1) \tag{2.1.15}$$

and $\boldsymbol{u}_{\sigma_j, u_j\to v} = (v, u_{j+1}, \dots, u_{j-1})$.

Proof. The approach is the same as that for the $\mathfrak{gl}_2$ case: we divide the problem into the 'wanted' and 'unwanted' terms, first calculating the wanted terms, then extending this to the unwanted terms using symmetry.

Consider (2.1.4). The *wanted term* in each exchange is the term in which the $a(v)$ operator retains its parameter, and it is clear that there will be a single wanted term after m exchanges. Additionally, by repeatedly applying the exchange relation, the unwanted terms may be categorised by the parameter u_i held by $a(\cdot)$ after all exchanges have been made. This may be summarised in the equation

$$a(v)\, B_{\boldsymbol{a}}(\boldsymbol{u}) = \left(\prod_{i=1}^{m} \frac{v-u_i+1}{v-u_i}\right) B_{\boldsymbol{a}}(\boldsymbol{u})\, a(v) + \sum_{j=1}^{m} U_{1,j}(v;\boldsymbol{u}).$$

where $U_{1,j}(v;\boldsymbol{u})$ denotes the unwanted terms containing $a(u_i)$ as the rightmost operator, or more specifically, $U_{1,j}(v;\boldsymbol{u})$ is of the form $Ba(u_j)$ for some product of operators $B \in (V'_{a_1})^* \otimes \cdots \otimes (V'_{a_m})^* \otimes \mathcal{B}((v^{-1}, u_1^{-1}, \dots, u_m^{-1}))$.

To find $U_{1,1}(v;\boldsymbol{u})$, we begin by acting on $B_{\boldsymbol{a}}(\boldsymbol{u})$ with $a(v)$. From (2.1.4), we have

$$a(v)\, B_{\boldsymbol{a}}(\boldsymbol{u}) = \left(\frac{v-u_1+1}{v-u_1} B_{a_1}(u_1)\, a(v) - \frac{1}{v-u_1} B_{a_1}(v)\, a(u_1)\right) B_{a_2}(u_2)\cdots B_{a_m}(u_m).$$

Now, moving $a(v)$ through the remaining creation operators, we note that the only contribution to

$U_{1,1}(v;\boldsymbol{u})$ will be from the second term in the above expression, in the instance when there are no further parameter swaps in the remaining commutations. Therefore,

$$U_{1,1}(v;\boldsymbol{u}) = -\frac{1}{v-u_1}\prod_{j=2}^{m}\frac{u_1-u_j+1}{u_1-u_j}\,B_{a_1}(v)\,B_{a_2}(u_2)\cdots B_{a_m}(u_m)\,a(u_1).$$

The $U_{1,j}(v;\boldsymbol{u})$ may be found by first applying a permutation to the parameters $\boldsymbol{u}$ via (2.1.12), then utilising the same argument as above. Indeed,

$$\begin{aligned} a(v)B_{\boldsymbol{a}}(\boldsymbol{u}) &= a(v)B_{\boldsymbol{a}}(\boldsymbol{u}_{\sigma_j})\check{R}'_{\boldsymbol{a}}[\sigma_j](\boldsymbol{u}) \\ &= \left(\prod_{i=1}^{m}\frac{v-u_{\sigma_j(i)}+1}{v-u_{\sigma_j(i)}}\right)B_{\boldsymbol{a}}(\boldsymbol{u}_{\sigma_j})\check{R}'_{\boldsymbol{a}}[\sigma_j](\boldsymbol{u})\,a(v)+\sum_{k=1}^{m}U_{1,k}(v;\boldsymbol{u}), \end{aligned}$$

and we find

$$U_{1,j}(v;\boldsymbol{u}) = -\frac{1}{v-u_j}\left(\prod_{k\neq j}\frac{u_j-u_k+1}{u_j-u_k}\right)B_{a_1}(v)\,B_{a_2}(u_{j+1})\cdots B_{a_m}(u_{j-1})\,\check{R}'_{\boldsymbol{a}}[\sigma_j](\boldsymbol{u})a(u_j).$$

Finally, we note that we may write this expression as a residue, so that the unwanted term resembles the wanted term:

$$U_{1,j}(v;\boldsymbol{u}) = -\frac{1}{v-u_j}\,B_{a_1}(v)\,B_{a_2}(u_{j+1})\cdots B_{a_m}(u_{j-1})\,\check{R}'_{\boldsymbol{a}}[\sigma_j](\boldsymbol{u})\,\operatorname*{Res}_{w\to u_j}\left[\left(\prod_{i=1}^{m}\frac{w-u_i+1}{w-u_i}\right)a(w)\right].$$

Putting this together with the wanted term, we arrive at expression (2.1.13).

For the $D_a(v)$ operator, we see that the wanted term from each exchange contains an R matrix which links the auxiliary spaces of the $D_a(v)$ and $B_{a_1}(u)$ operators. Since this R matrix acts only on those two spaces, it commutes with any creation operators to the right of it and may be moved to the right of the expression before applying (2.1.5) to the second creation operator. Therefore, taking the trace after m exchanges we have

$$\operatorname{tr}_a D_a(v)\,B_{\boldsymbol{a}}(\boldsymbol{u}) = B_{\boldsymbol{a}}(\boldsymbol{u})\operatorname{tr}_a\left(D_a(v)\,R'_{aa_m}(v-u_m)\cdots R'_{aa_1}(v-u_1)\right)+\sum_{j=1}^{m}U_{2,j}(v;\boldsymbol{u}),$$

where now each $U_{2,j}(v;\boldsymbol{u})$ may be written $\sum_{k,l=1}^{n-1}B_{kl}\,d_{kl}(u_j)$ for some $B_{kl}\in(V'_{a_1})^*\otimes\cdots\otimes(V'_{a_m})^*\otimes\mathcal{B}((v^{-1},u_1^{-1},\ldots,u_m^{-1}))$.

In order to find the unwanted terms, we first note that the unwanted term from a single exchange may be written as follows, working from (2.1.5) and taking the trace:

$$\operatorname{tr}_a D_a(v)B_{a_1}(u) = B_{a_1}(u)\,\operatorname{tr}_a\left[D_a(v)\,R'_{aa_1}(v-u)\right]-\frac{1}{v-u}B_{a_1}(v)\,\operatorname*{Res}_{w\to u}\,\operatorname{tr}_a\left[D_a(w)R'_{aa_1}(w-u)\right],$$

as $P=-\operatorname*{Res}_{w\to u}R(w-u)$. In this form we see that the unwanted term is the residue of the wanted term, and can proceed with the argument that was used for $a(v)$. Acting now on $B_{\boldsymbol{a}}(\boldsymbol{u})$, we obtain $U_{2,1}(v;\boldsymbol{u})$ by starting with the unwanted term from the first exchange given above and taking the wanted term in each subsequent exchange, yielding

$$U_{2,1}(v;\boldsymbol{u}) = \frac{1}{v-u_1} B_{a_1}(v)\, B_{a_2}(u_2) \cdots B_{a_m}(u_m) \operatorname*{Res}_{w\to u_1} \operatorname{tr}_a T'_{a;\boldsymbol{a}}(w;\boldsymbol{u}).$$

with $T'_{a;\boldsymbol{a}}(w;\boldsymbol{u})$ as given in (2.1.15). Then using the same permutation of the parameters $\boldsymbol{u}$ as above, we find

$$U_{2,j}(v;\boldsymbol{u}) = -\frac{1}{v-u_j} B_{\boldsymbol{a}}(\boldsymbol{u}_{\sigma_j, u_j\to v}) \operatorname*{Res}_{w\to u_1} \operatorname{tr}_a T'_{a;\boldsymbol{a}}(w;\boldsymbol{u}_{\sigma_j})\, \check{R}'_{\boldsymbol{a}}[\sigma_j](\boldsymbol{u}),$$

as required. □

Here note that the matrix that appears on the r.h.s of the exchange relations for $D_a(v)$ is not $D_a(v)$ itself but in fact the matrix

$$T'_{a;\boldsymbol{a}}(v;\boldsymbol{u}) := D_a(v)\, R'_{aa_m}(v-u_m) \cdots R'_{aa_1}(v-u_1). \tag{2.1.16}$$

We will refer to this matrix as the *nested monodromy matrix*, and proceed to investigate its properties.

Lemma 2.1.4. *The nested monodromy matrix satisfies the RTT relation,*

$$R'_{ab}(v-w)\, T'_{a;\boldsymbol{a}}(v;\boldsymbol{u})\, T'_{b;\boldsymbol{a}}(w;\boldsymbol{u}) = T'_{b;\boldsymbol{a}}(w;\boldsymbol{u})\, T'_{a;\boldsymbol{a}}(v;\boldsymbol{u})\, R'_{ab}(v-w).$$

Proof. Starting from the l.h.s. of the equation and using the definition (2.1.16) of $T'_{a;\boldsymbol{a}}(v;\boldsymbol{u})$,

$$\begin{aligned}
&R'_{ab}(v-w)\, T'_{a;\boldsymbol{a}}(v;\boldsymbol{u})\, T'_{b;\boldsymbol{a}}(w;\boldsymbol{u}) \\
&= R'_{ab}(v-w)\, D_a(v)\, R'_{aa_m}(v-u_m) \cdots R'_{aa_1}(v-u_1)\, D_b(w)\, R'_{ba_m}(w-u_m) \cdots R'_{ba_1}(w-u_1) \\
&= R'_{ab}(v-w)\, D_a(v) D_b(w)\, R'_{aa_m}(v-u_m)\, R'_{ba_m}(w-u_m) \cdots R'_{aa_1}(v-u_1) R'_{ba_1}(w-u_1) \\
&= D_b(w) D_a(v) R'_{ab}(v-w)\, R'_{aa_m}(v-u_m)\, R'_{ba_m}(w-u_m) \cdots R'_{aa_1}(v-u_1) R'_{ba_1}(w-u_1) \text{ by (2.1.7)} \\
&= D_b(w) D_a(v)\, R'_{ba_m}(w-u_m)\, R'_{aa_m}(v-u_m) \cdots R'_{ba_1}(w-u_1) R'_{aa_1}(v-u_1) R'_{ab}(v-w) \quad \text{by YBE} \\
&= T'_{b;\boldsymbol{a}}(w;\boldsymbol{u})\, T'_{a;\boldsymbol{a}}(v;\boldsymbol{u})\, R'_{ab}(v-w).
\end{aligned}$$

□

By the above Lemma, the matrix $T'_a(v;\boldsymbol{u})$ is a homomorphic image of the generating matrix $T'_{a;\boldsymbol{a}}(v)$ of $Y(\mathfrak{gl}_{n-1})$.

Another important property concerns the parameters $\boldsymbol{u}$ which appear in the nested monodromy matrix.

Lemma 2.1.5. *Matrix elements* $\left[T'_{a;\boldsymbol{a}}(v;\boldsymbol{u})\right]_{jk}$ *of* $T'_{a;\boldsymbol{a}}(v;\boldsymbol{u})$ *transform under the action of* S_m *as:*

$$\check{R}'_{\boldsymbol{a}}[\sigma](\boldsymbol{u}) \left[T'_{a;\boldsymbol{a}}(v;\boldsymbol{u})\right]_{jk} = \left[T'_{a;\boldsymbol{a}}(v;\boldsymbol{u}_\sigma)\right]_{jk} \check{R}'_{\boldsymbol{a}}[\sigma](\boldsymbol{u}),$$

for any $\sigma \in S_m$.

Proof. It is sufficient to show that this relation holds for transpositions, as these generate S_m. Moving $\check{R}'_{a_i a_{i+1}}(u_i - u_{i+1})$ from left to right through each of the R-matrices in the definition (2.1.16), the R-matrices with which it does not commute will undergo parameter exchange $u_i \leftrightarrow u_{i+1}$ due to the (braided) Yang-Baxter equation:

$$\check{R}'_{a_i a_{i+1}}(u_i - u_{i+1}) R'_{a a_{i+1}}(v - u_{i+1}) R'_{a a_i}(v - u_i) = R'_{a a_{i+1}}(v - u_i) R'_{a a_i}(v - u_{i+1}) \check{R}'_{a_i a_{i+1}}(u_i - u_{i+1}).$$

Thus we obtain

$$\check{R}'_{a_i a_{i+1}}(u_i - u_{i+1}) \left[T'_{a;\boldsymbol{a}}(v;\boldsymbol{u})\right]_{jk} = \left[T'_{a;\boldsymbol{a}}(v;\boldsymbol{u}_{i\leftrightarrow i+1})\right]_{jk} \check{R}'_{a_i a_{i+1}}(u_i - u_{i+1}).$$

The required identity is now immediate. $\square$

We have established that the nested monodromy matrix satisfies the RTT relation with the $\mathfrak{gl}_{n-1}$ R-matrix, and so defines a representation of $Y(\mathfrak{gl}_{n-1})$. We will refer to the associated representation space as the *nested vacuum sector*, as it will play the role of the vacuum vector on which the creation operators $B(u)$ act.

Denote by $L(\lambda^{(i)})^0_{c_i}$ the subspace of the $Y(\mathfrak{gl}_n)$-evaluation module $L(\lambda^{(i)})_{c_i}$ consisting of vectors annihilated by all operators $c_j(u)$, namely

$$L(\lambda^{(i)})^0_{c_i} := \{\zeta \in L(\lambda^{(i)})_{c_i} \,:\, c_j(u)\zeta = 0 \text{ for } 1 \le j \le n-1\}.$$

This subspace corresponds to the natural embedding $\mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n$ and is an irreducible lowest weight $Y(\mathfrak{gl}_{n-1})$-module with the lowest weight given by

$$\lambda_i(u)^0 = \lambda_{i+1}(u) \quad \text{for} \quad 1 \le i \le n-1 \tag{2.1.17}$$

and $\lambda_i(u)$ defined in (2.1.2).

We define the *vacuum sector* $L^0 \subset L$ as the tensor product of these subspaces,

$$L^0 := L(\lambda^{(1)})^0_{c_1} \otimes L(\lambda^{(2)})^0_{c_2} \otimes \ldots \otimes L(\lambda^{(\ell)})^0_{c_\ell}.$$

By our initial assumption, the space L is an irreducible $Y(\mathfrak{gl}_n)$-module. Then, by Lemma 6.2.2 and Theorem 6.5.8 in [Mo07], the space L^0 is an irreducible $Y(\mathfrak{gl}_{n-1})$-module. In particular, the space L^0 is annihilated by all operators $c_i(u)$,

$$L^0 = \{\zeta \in L \,:\, c_i(u)\cdot\zeta = 0 \;\text{ for }\; 1 \le i \le n-1\},$$

and is stable under the action of the operators $d_{ij}(u)$ for $1 \le i, j \le n-1$, see (2.1.8).

Each auxiliary space V'_{a_i} is a vector representation of the Lie algebra $\mathfrak{gl}_{n-1}$ of weight $\lambda' = (1, 0, \ldots, 0)$ and may now be viewed as an evaluation module $L(\lambda')_{u_i}$ of $Y(\mathfrak{gl}_{n-1})$ with the lowest weight given by

$$\lambda_1'(u) = \frac{u - u_i - 1}{u - u_i} \quad \text{and} \quad \lambda_j'(u) = 1 \quad \text{for} \quad 2 \le j \le n-1. \tag{2.1.18}$$

In particular, the generating matrix $T_a'(u)$ of $Y(\mathfrak{gl}_{n-1})$ acts on $L(\lambda')_{u_i}$ as $R'_{aa_i}(u - u_i)$.

We have now all the necessary ingredients to define the *nested vacuum sector*

$$L' := L^0 \otimes V'_{a_m} \otimes \cdots \otimes V'_{a_1}. \tag{2.1.19}$$

Proposition 2.1.6. *Let $T'(v)$ denote the generating matrix of $Y(\mathfrak{gl}_{n-1})$. Then the map*

$$Y(\mathfrak{gl}_{n-1}) \to Y(\mathfrak{gl}_n) \otimes \mathrm{End}(V'_{a_m} \otimes \cdots \otimes V'_{a_1}), \quad T'(v) \mapsto T'(v; \boldsymbol{u}) \tag{2.1.20}$$

is a homomorphism of algebras. Moreover, it equips the space L' with a structure of a lowest weight $Y(\mathfrak{gl}_{n-1})$-module with the lowest weight given by

$$\begin{aligned} \lambda_1'(v; \boldsymbol{u}) &= \prod_{j=1}^{\ell} \frac{v - \lambda_2^{(j)} - c_j - 1}{v - \lambda_2^{(j)} - c_j} \prod_{k=1}^{m} \frac{v - u_k - 1}{v - u_k} \quad \textit{and} \\ \lambda_i'(v; \boldsymbol{u}) &= \prod_{j=1}^{\ell} \frac{v - \lambda_{i+1}^{(j)} - c_j - 1}{v - \lambda_{i+1}^{(j)} - c_j} \quad \textit{for} \quad 2 \le i \le n-1. \end{aligned} \tag{2.1.21}$$

Proof. The homomorphism property follows from Lemma 2.1.4. We already know that L^0 is an irreducible $Y(\mathfrak{gl}_{n-1})$-module. It follows from (2.1.16) and (2.1.19) that the space L' is stable under the action of $T'_{a;\boldsymbol{a}}(v; \boldsymbol{u})$. Thus the map (2.1.20) equips the space L' with a structure of $Y(\mathfrak{gl}_{n-1})$-module with each tensorand a lowest weight $Y(\mathfrak{gl}_{n-1})$-module. The lowest vector is

$$\eta = \eta_1 \otimes \cdots \otimes \eta_\ell \otimes e_1' \otimes \cdots \otimes e_1', \tag{2.1.22}$$

where each η_i is a lowest vector of $L(\lambda^{(i)})^0_{c_i}$ for $1 \le i \le \ell$ and each e_1' is a lowest vector of V_{a_i} for $1 \le i \le m$ (viewed as an evaluation module $L(\lambda')_{u_i}$). Finally, acting with $[T'_{a;\boldsymbol{a}}(v; \boldsymbol{u})]_{ii}$ on η for $1 \le i \le n$ and using (2.1.17) and (2.1.18) yields (2.1.21). □

Lemma 2.1.7. *For any vector $\zeta \in L'$ we have that $a(u) \cdot \zeta = \lambda_1(u)\zeta$, where $\lambda_1(u)$ is defined by (2.1.2).*

Proof. By Proposition 2.1.6 we know that $L' = Y(\mathfrak{gl}_{n-1})\eta$ for η defined in (2.1.22) and $c_i(u) \cdot L' = 0$. Using (2.1.9) and definition of $t'_{ij}(v; \boldsymbol{u})$, we find that $[a(u), t'_{ij}(v; \boldsymbol{u})] \cdot \zeta = 0$ for any $1 \le i, j \le n$. Hence it is enough to act with $a(u)$ on the lowest vector η, which yields the required result. □

2.1.3 Nested algebraic Bethe ansatz

Recall the definition of the full quantum space (2.1.1) and the nested vacuum sector (2.1.19). Let $\Phi' \in L'$. We will refer to this as the *nested Bethe vector*, and we will impose additional properties on it in what follows. The ansatz for the eigenvector of the transfer matrix, the *Bethe vector*, is then

$$\Phi(\boldsymbol{u}) := B_{\boldsymbol{a}}(\boldsymbol{u}) \cdot \Phi' \in L.$$

Since L is a finite dimensional vector space, the parameters $\boldsymbol{u}$ can be evaluated to non-zero complex numbers, hence from now on we will assume that $\boldsymbol{u} \in \mathbb{C}^m$ is an m-tuple of non-zero complex numbers.

To find the conditions for which this is an eigenvector of the transfer matrix, we act with $t(v) = a(v) + \operatorname{tr}_a D_a(v)$, and use Lemma 2.1.3 to move through the creation operators. So

$$\begin{aligned}
&\big(a(v) + \operatorname{tr}_{\boldsymbol{a}} D_a(v)\big) B_{\boldsymbol{a}}(\boldsymbol{u}) \cdot \Phi' \\
&= B_{\boldsymbol{a}}(\boldsymbol{u}) \left[\left(\prod_{i=1}^{m} \frac{v - u_i + 1}{v - u_i} \right) a(v) + \operatorname{tr}_a T'_{a;\boldsymbol{a}}(v;\boldsymbol{u}) \right] \cdot \Phi' \\
&\quad - \sum_{j=1}^{m} \frac{1}{v - u_j} B_{\boldsymbol{a}}(\boldsymbol{u}_{\sigma_j, u_j \to v}) \operatorname*{Res}_{w \to u_j} \left[\left(\prod_{i=1}^{m} \frac{w - u_i + 1}{w - u_i} \right) a(w) + \operatorname{tr}_a T'_{a;\boldsymbol{a}}(w; \boldsymbol{u}_{\sigma_j}) \right] \check{R}'_{\boldsymbol{a}}[\sigma_j](\boldsymbol{u}) \cdot \Phi'.
\end{aligned} \tag{2.1.23}$$

For $\Phi(\boldsymbol{u})$ to be an eigenvector of $t(v)$ for all v, we require that the wanted term acts as a scalar on Φ', and that the unwanted terms all vanish on Φ'. Focussing first on the wanted term, by Lemma 2.1.7, $a(v) \cdot \Phi' = \lambda_1(v) \Phi'$. Therefore, we require that Φ' be an eigenvector of the nested transfer matrix $t'_{;\boldsymbol{a}}(v;\boldsymbol{u}) := \operatorname{tr}_a T'_{a;\boldsymbol{a}}(v;\boldsymbol{u})$ for all v, that is,

$$t'_{;\boldsymbol{a}}(v;\boldsymbol{u}) \cdot \Phi' = \Gamma'(v;\boldsymbol{u}) \Phi', \tag{2.1.24}$$

for some scalar $\Gamma'(v;\boldsymbol{u})$. Under this assumption, the eigenvalue of the transfer matrix $t(v)$ will then be

$$\Gamma(v;\boldsymbol{u}) = \lambda_1(v) \prod_{i=1}^{m} \frac{v - u_i + 1}{v - u_i} + \Gamma'(v;\boldsymbol{u}). \tag{2.1.25}$$

This condition on Φ' presents an equivalent diagonalisation problem to the one defined initially for $t(v)$, the auxiliary spaces $a_1, \ldots, a_m$ acting as additional spin chain sites in the vector representation, with parameter shifts given by the corresponding $u_1, \ldots, u_m$. Crucially, the underlying algebra has changed from $Y(\mathfrak{gl}_n)$ to $Y(\mathfrak{gl}_{n-1})$, with monodromy matrix $T_{a;\boldsymbol{a}}(v;\boldsymbol{u})$ and, as such, we may continue this argument inductively until arriving at the known $Y(\mathfrak{gl}_2)$ case.

For example, constructing the ansatz for the nested Bethe vector, we fix $m' \in \mathbb{N}$ and introduce an m'-tuple $\boldsymbol{u}' = (u'_1, \ldots, u'_{m'})$ of distinct complex parameters, so that

$$\Phi' = \Phi'(\boldsymbol{u}';\boldsymbol{u}) = B'_{a'_1}(u'_1;\boldsymbol{u}) \cdots B'_{a'_{m'}}(u'_{m'};\boldsymbol{u}) \cdot \Phi'',$$

where, upon decomposing the nested monodromy matrix $T'_{a;\boldsymbol{a}}(v, \boldsymbol{u})$ in the same way as we did for $T_a(v)$,

$$\Phi'' \in L'^0 \otimes V''_{a'_{m'}} \otimes \cdots \otimes V''_{a'_1}.$$

Here L'^0 is the vacuum sector of L' defined analogously to that of L, and each $V''_{a'_i}$ is a $\mathfrak{gl}_{n-2}$-module of weight $\lambda'' = (1, 0, \ldots, 0)$. Repeating this process, we reduce the problem to a $Y(\mathfrak{gl}_2)$-system, which may be solved using using an argument of the type given in Chapter 1.

It therefore remains to show that the unwanted terms can be made to vanish on the nested Bethe vector. Since the creation operators within each unwanted term contain a unique set of parameters, we assume that the unwanted terms are each linearly independent, and demand the potentially stronger condition on Φ' that each term vanishes independently.

Consider again the expression for the unwanted terms acting on Φ' in (2.1.23). As a consequence of Lemma 2.1.5 we may commute the j^{th} unwanted term in square brackets with the $\check{R}'_{\boldsymbol{a}}[\sigma_j](\boldsymbol{u})$, which reverses the parameter permutation in the nested monodromy matrix. The expression on this line then becomes

$$-\sum_{j=1}^{m} \frac{1}{v-u_j} B_{\boldsymbol{a}}(\boldsymbol{u}_{\sigma_j, u_j \to v}) \check{R}'_{\boldsymbol{a}}[\sigma_j](\boldsymbol{u}) \operatorname*{Res}_{w \to u_j} \left[\left(\prod_{i=1}^{m} \frac{w-u_i+1}{w-u_i} \right) a(w) + \operatorname{tr}_a T'_{a;\boldsymbol{a}}(w; \boldsymbol{u}) \right] \cdot \Phi'.$$

Each term in this sum is now merely a residue of the wanted term acting on Φ', which we have assumed acts diagonally (2.1.24). Therefore, this expression becomes

$$-\sum_{j=1}^{m} \frac{1}{v-u_j} B_{\boldsymbol{a}}(\boldsymbol{u}_{\sigma_j, u_j \to v}) \check{R}'_{\boldsymbol{a}}[\sigma_j](\boldsymbol{u}) \operatorname*{Res}_{w \to u_j} \left[\left(\prod_{i=1}^{m} \frac{w-u_i+1}{w-u_i} \right) \lambda_1(w) + \Gamma'(w; \boldsymbol{u}) \right] \cdot \Phi'.$$

By setting each summand to zero, we obtain a sufficient condition for the unwanted terms to vanish

$$\operatorname*{Res}_{w \to u_j} \left[\left(\prod_{i=1}^{m} \frac{w-u_i+1}{w-u_i} \right) \lambda_1(w) + \Gamma'(w; \boldsymbol{u}) \right] = 0$$

Or, in other words,

$$\operatorname*{Res}_{w \to u_j} \Gamma(w; \boldsymbol{u}) = 0 \quad \text{for} \quad 1 \le j \le m. \tag{2.1.26}$$

These are the *Bethe equations* for $\boldsymbol{u}$.

2.1.4 End of recursion

Upon reducing to the residual $Y(\mathfrak{gl}_2)$-system, we have the familiar 2×2 monodromy matrix

$$T_a^{(n-2)}(v) = \begin{pmatrix} a^{(n-2)}(v) & b^{(n-2)}(v) \\ c^{(n-2)}(v) & d^{(n-2)}(v) \end{pmatrix}.$$

Dependence on parameters $\boldsymbol{u}, \boldsymbol{u}', \ldots, \boldsymbol{u}^{(n-3)}$ has been suppressed. The RTT relation yields the relations

$$\begin{aligned}
a^{(n-2)}(v)\, b^{(n-2)}(u) &= \frac{v-u+1}{v-u} b^{(n-2)}(u)\, a^{(n-2)}(v) - \frac{1}{v-u} b^{(n-2)}(v)\, a^{(n-2)}(u), \\
d^{(n-2)}(v)\, b^{(n-2)}(u) &= \frac{v-u-1}{v-u} b^{(n-2)}(u)\, d^{(n-2)}(v) + \frac{1}{v-u} b^{(n-2)}(v)\, d^{(n-2)}(u), \\
[b^{(n-2)}(v), b^{(n-2)}(u)] &= 0.
\end{aligned}$$

The Bethe vector with $m^{(n-2)}$ excitations is

$$\Phi^{(n-2)}(\boldsymbol{u}) = b^{(n-2)}(u_1^{(n-2)}) \cdots b^{(n-2)}(u_{m^{(n-2)}}^{(n-2)}) \cdot \eta^{(n-2)},$$

where $\eta^{(n-2)}$ is a lowest vector of the nested vacuum sector $L^{(n-2)}$. The associated eigenvalue of the transfer matrix $t^{(n-2)}(v)$ is

$$\begin{aligned}
\Gamma^{(n-2)}(v; \boldsymbol{u}, \ldots, \boldsymbol{u}^{(n-2)}) = \lambda_1^{(n-2)}(v; \boldsymbol{u}, \ldots, \boldsymbol{u}^{(n-3)}) \prod_{i=1}^{m^{(n-2)}} \frac{v - u^{(n-2)} + 1}{v - u^{(n-2)}} \\
+ \lambda_2^{(n-2)}(v; \boldsymbol{u}, \ldots, \boldsymbol{u}^{(n-3)}) \prod_{i=1}^{m^{(n-2)}} \frac{v - u^{(n-2)} - 1}{v - u_i^{(n-2)}},
\end{aligned}$$

provided the $\boldsymbol{u}^{(n-2)}$ satisfy the Bethe equations

$$\operatorname*{Res}_{w \to u_j^{(n-2)}} \Gamma^{(n-2)}(w; \boldsymbol{u}, \ldots \boldsymbol{u}^{(n-2)}) = 0 \quad \text{for} \quad 1 \leq j \leq m^{(n-2)}.$$

2.1.5 Full expressions for eigenvalues and Bethe equations

In this section, we unpack the recursion steps to give the explicit expressions for the eigenvalues of the transfer matrix in terms of the parameters of the $Y(\mathfrak{gl}_n)$ system. In order to match the notation used in the Bethe ansatz for the $Y_\rho^{\pm}(\mathfrak{gl}_{2n})$ chain in Section 2.2, we begin by relabelling the spectral parameters as follows. For the initial step, relabel parameters $u_i \to u_i^{(1)}$ and excitation number $m \to m^{(1)}$, and for subsequent levels of nesting $u_i^{(k)} \to u_i^{(k+1)}$ and $m^{(k)} \to m^{(k+1)}$. We use Proposition 2.1.6 to rewrite the weights $\lambda_1^{(k)}(v; \boldsymbol{u}, \ldots, \boldsymbol{u}^{(k-1)})$ of the nested system in terms of the weights of the initial $Y(\mathfrak{gl}_n)$-system,

$$\begin{aligned}
\lambda_1^{(k)}(v; \boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(k)}) = \lambda_{k+1}(v) \prod_{i=1}^{m^{(k)}} \frac{v - u_j^{(k)} - 1}{v - u_j^{(k)}} \quad \text{for} \quad 1 \leq k \leq n-1 \quad \text{and} \\
\lambda_l^{(k)}(v; \boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(k)}) = \lambda_{k+l}(v) \quad \text{for} \quad l > 1, \quad 1 \leq k \leq n-l.
\end{aligned}$$

From the recursion relation in (2.1.25), a general expression can be found for the eigenvalue $\Gamma^{(k)}(v;\boldsymbol{u}^{(1)},\ldots,\boldsymbol{u}^{(n-1)})$, for $1\le k\le n-2$:

$$\begin{aligned}
&\Gamma^{(k)}(v;\boldsymbol{u}^{(1)},\ldots,\boldsymbol{u}^{(n-1)})\\
&\quad=\lambda_2^{(n-2)}(v;\boldsymbol{u}^{(1)},\ldots,\boldsymbol{u}^{(n-2)})\prod_{i=1}^{m^{(n-1)}}\frac{v-u_i^{(n-1)}-1}{v-u_i^{(n-1)}}+\sum_{l=k}^{n-2}\lambda_1^{(l)}(v;\boldsymbol{u}^{(1)},\ldots,\boldsymbol{u}^{(l)})\prod_{i=1}^{m^{(l+1)}}\frac{v-u^{(l+1)}+1}{v-u^{(l+1)}}\\
&\quad=\lambda_n(v)\prod_{i=1}^{m^{(n-1)}}\frac{v-u_i^{(n-1)}-1}{v-u_i^{(n-1)}}+\sum_{l=k}^{n-2}\lambda_{l+1}(v)\prod_{i=1}^{m^{(l)}}\frac{v-u_i^{(l)}-1}{v-u_i^{(l)}}\prod_{i=1}^{m^{(l+1)}}\frac{v-u_i^{(l+1)}+1}{v-u_i^{(l+1)}}.
\end{aligned}$$

We have thus shown the following.

Theorem 2.1.8. *The eigenvalues of the Bethe vectors for a $Y(\mathfrak{gl}_n)$-system are given by*

$$\begin{aligned}
\Gamma(v)&=\lambda_1(v)\prod_{i=1}^{m^{(1)}}\frac{v-u_i^{(1)}+1}{v-u_i^{(1)}}+\lambda_n(v)\prod_{i=1}^{m^{(n-1)}}\frac{v-u_i^{(n-1)}-1}{v-u_i^{(n-1)}}\\
&\quad+\sum_{l=1}^{n-2}\lambda_{l+1}(v)\prod_{i=1}^{m^{(l)}}\frac{v-u_i^{(l)}-1}{v-u_i^{(l)}}\cdot\prod_{i=1}^{m^{(l+1)}}\frac{v-u_i^{(l+1)}+1}{v-u_i^{(l+1)}}.
\end{aligned}\tag{2.1.27}$$

Recall also the Bethe equations (2.1.26) satisfied by parameters $u_j^{(k)}$. In fact, comparing the above two expressions, we note that equivalent Bethe equations can be obtained by demanding instead that the residue of the full eigenvalue $\Gamma(v)$ vanishes at each $u_j^{(k)}$ for $1\le k\le n-1$, $1\le j\le m^{(k)}$. This is exactly the condition that the eigenvalue of the transfer matrix is analytic. We may now evaluate the residue to obtain the Bethe equations in terms of $\lambda_k(v)$ with $1\le k\le n$ leading to the following statement.

Theorem 2.1.9. *The Bethe equations for a $Y(\mathfrak{gl}_n)$-system are*

$$\begin{aligned}
\frac{\lambda_k(u_j^{(k)})}{\lambda_{k+1}(u_j^{(k)})}&=\prod_{i=1}^{m^{(k-1)}}\frac{u_j^{(k)}-u_i^{(k-1)}}{u_j^{(k)}-u_i^{(k-1)}-1}\cdot\prod_{i\neq j}\frac{u_j^{(k)}-u_i^{(k)}-1}{u_j^{(k)}-u_i^{(k)}+1}\cdot\prod_{i=1}^{m^{(k+1)}}\frac{u_j^{(k)}-u_i^{(k+1)}+1}{u_j^{(k)}-u_i^{(k+1)}},\\
\frac{\lambda_1(u_j^{(1)})}{\lambda_2(u_j^{(1)})}&=\prod_{i\neq j}\frac{u_j^{(1)}-u_i^{(1)}-1}{u_j^{(1)}-u_i^{(1)}+1}\cdot\prod_{i=1}^{m^{(2)}}\frac{u_j^{(1)}-u_i^{(2)}+1}{u_j^{(1)}-u_i^{(2)}},\\
\frac{\lambda_{n-1}(u_j^{(n-1)})}{\lambda_n(u_j^{(n-1)})}&=\prod_{i=1}^{m^{(n-2)}}\frac{u_j^{(n-1)}-u_i^{(n-2)}}{u_j^{(n-1)}-u_i^{(n-2)}-1}\cdot\prod_{i\neq j}\frac{u_j^{(n-1)}-u_i^{(n-1)}-1}{u_j^{(n-1)}-u_i^{(n-1)}+1},
\end{aligned}\tag{2.1.28}$$

for $1\le k\le n-1$ and $1\le j\le m^{(k)}$.

Finally, we end this section with an expression for the eigenvector. Although we have constructed the eigenvector recursively, it was shown in [TV13],[BR08] that the $\mathfrak{gl}_n$ Bethe vector may instead be written in a closed form in terms of the original $Y(\mathfrak{gl}_n)$ generating matrix and R-matrix.

$$\Phi(\boldsymbol{u}^{(1)},\dots,\boldsymbol{u}^{(n-1)})$$
$$= \operatorname{tr}_{\overline{V}}\Bigg[\Bigg(\prod_{k=1}^{n-1}\prod_{i=1}^{m^{(k)}} T_{a_i^k}(u_i^{(k)})\Bigg)\Bigg(\prod_{k=2}^{n-1}\prod_{l=1}^{k-1}\prod_{i=1}^{m^{(k)}}\prod_{j=m^{(l)}}^{1} R_{a_i^k a_j^l}(u_i^{(k)}-u_j^{(l)})\Bigg) \times (e_{21})^{\otimes m^{(1)}}\otimes\cdots\otimes(e_{n,n-1})^{\otimes m^{(n-1)}}\Bigg]\cdot\eta, \tag{2.1.29}$$

where the trace is taken over the space $\overline{V} := V_{a_1^1}\otimes\cdots\otimes V_{a^{n-1}_{m^{(n-1)}}} \cong (\mathbb{C}^n)^{\otimes\overline{m}}$ with $\overline{m} = \sum_{i=1}^{n-1} m^{(i)}$ and η is the lowest weight vector for L.

2.2 Nested algebraic Bethe ansatz for the even twisted Yangian spin chain

In this section, we give the nested algebraic Bethe ansatz for an even twisted Yangian spin chain. We first review the twisted Yangian algebra and its representation theory, allowing us to define a spin chain as a twisted Yangian representation. We then proceed to construct eigenvectors for a transfer matrix which acts on this system using the nested algebraic Bethe ansatz.

2.2.1 The Ol'shanskii twisted Yangian

We begin by defining the Ol'shanskii twisted Yangian, following [Ol92], [Mo07]. First, recall the transpose defined in (1.2.20). Recall also the notation $\pm$ and $\mp$ with upper and lower signs referring to the orthogonal and sympelctic cases respectively.

Definition 2.2.1. *Let $\rho\in\mathbb{C}$. The twisted Yangian $Y_\rho^{\pm}(\mathfrak{gl}_{2n})$ is the subalgebra of $Y(\mathfrak{gl}_{2n})$ generated by the coefficients of the entries of the matrix*

$$S(u) = T(u)T^t(-u-\rho). \tag{2.2.1}$$

The 'ρ-shifted' twisted Yangian defined above is isomorphic to the usual one studied in [Mo07]. The isomorphism is provided by the mapping $S(u)\mapsto S(u+\rho/2)$. For the purposes of the nested algebraic Bethe ansatz, it will not be necessary to fix a particular value of ρ, and it will remain a free parameter. However, it may be necessary to fix a particular value of ρ in order to obtain a local interaction Hamiltonian from the resulting transfer matrix; see Remark 2.2.17 for more details.

Note that the construction (2.2.1) differs from the one described in (1.3.1) as it physically represents a reflective boundary which turns particles into antiparticles and vice versa. As such, instead of the reflection equation the matrix $S(u)$ satisfies the twisted reflection equation

$$R_{12}(u-v)\,S_1(u)\,R_{12}^t(-u-v-\rho)\,S_2(v) = S_2(v)\,R_{12}^t(-u-v-\rho)\,S_1(u)\,R_{12}(u-v), \tag{2.2.2}$$

which is a simple consequence of the RTT relation.

Additionally it satisfies the *symmetry relation*

$$S^t(-u-\rho) = S(u) \pm \frac{S(u) - S(-u-\rho)}{2u+\rho}\,, \tag{2.2.3}$$

which can be shown by taking the transpose of $S(u)$ and using the $Y(\mathfrak{gl}_n)$ relations to put the matrix back together. Indeed, if $S(u)$ were instead constructed from a matrix of commutative polynomials we would simply have the first term on the right hand side; the extra terms are a consequence of the noncommutative nature of the Yangian. The fact that this relation is linear will be useful for the nested algebraic Bethe ansatz.

The above two relations are in fact the defining relations of $Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})$, in the sense that this subalgebra is isomorphic to an algebra defined by the above relations only. Their form in terms of matrix elements $s_{ij}(u)$ of $S(u)$, for $\rho = 0$, can be found in (2.4) and (2.5) of [Ol92] (note that indices i, j, k, l are indexed by $-n, -n+1, \dots, n-1, n$ in *loc. cit.*); also see Section 4.1 in [Mo07].

This definition of the twisted Yangian may be seen as a quantum analogue of the classical definition of $\mathfrak{g}_{2n}$ as the invariant subalgebra of $\mathfrak{gl}_{2n}$ with respect to the automorphism $E \mapsto -E^t$. In other words, $Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})$ is the rational quantum version of symmetric pairs AI(a) and AII. Denoting the coefficients as $s_{ij}(u) = \sum_{r\geq 0} s^{(r)}_{ij} u^{-r}$, the map $F_{ij} \mapsto -s^{(1)}_{ji}$ defines an embedding $U(\mathfrak{g}_{2n}) \hookrightarrow Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})$, with the twisted reflection equation and symmetry relation implying (1.2.18) and (1.2.19) respectively. In fact, for this algebra in particular, the map

$$s_{ij}(u) \mapsto \delta_{ij} - F_{ji}(u + (\rho \pm 1)/2)^{-1}$$

defines the *evaluation homomorphism* $ev^{\pm} : Y^{\pm}_{\rho}(\mathfrak{gl}_{2n}) \to U(\mathfrak{g}_{2n})$. This is in contrast to $X(\mathfrak{g}_N)$, which cannot possess such an evaluation homomorphism.

Recall now the coproduct for $Y(\mathfrak{gl}_{2n})$, defined by (1.2.11). The coproduct is itself an algebra homomorphism, and the action on the twisted Yangian subalgebra generated by $S(u)$ can be shown to obey

$$\Delta : S(u) \mapsto (T(u)\otimes 1)(1\otimes S(u))(T^t(-u-\rho)\otimes 1) \in \mathrm{End}(\mathbb{C}^{2n})\otimes Y(\mathfrak{gl}_{2n})\otimes Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})[[u^{-1}]]. \tag{2.2.4}$$

We see that, rather than mapping to the tensor square of the twisted Yangian, the subalgebra is preserved only in the right tensor space in the image of the coproduct, that is, $\Delta : Y^{\pm}_{\rho}(\mathfrak{gl}_{2n}) \to Y(\mathfrak{gl}_{2n}) \otimes Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})$. This property makes the twisted Yangian a *left coideal* subalgebra of $Y(\mathfrak{gl}_{2n})$.

This also has a clear physical interpretation: the $T(u)$ and $T^t(-u-\rho)$ are the interaction of a test particle with the 'bulk' of the spin chain before and after reflecting off the boundary respectively, and $S(u)$ is the interaction with the boundary itself.

We now turn to representation theory of $Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})$. As in the case of $Y(\mathfrak{gl}_n)$, we will be interested in the lowest weight representations.

Definition 2.2.2. *A representation V of $Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})$ is called a lowest weight representation if there* *exists a nonzero vector $\xi \in V$ such that $V = Y_\rho^\pm(\mathfrak{gl}_{2n})\xi$ and*

$$\begin{aligned} s_{ij}(u)\xi &= 0 && \text{for} \quad 1 \le j < i \le 2n \quad \text{and} \\ s_{ii}(u)\xi &= \mu_i(u)\xi && \text{for} \quad 1 \le i \le n, \end{aligned}$$

where $\mu_i(u)$ are formal power series in u^{-1} with constant terms equal to 1. The vector ξ is called the lowest weight vector of V, and the n-tuple $\mu(u) = (\mu_1(u), \ldots, \mu_n(u))$ is called the lowest weight of V.

Note that ξ is also an eigenvector for the action of $s_{ii}(u)$ with $n+1 \le i \le 2n$. Indeed, the symmetry relation (2.2.3) implies that

$$s_{2n-i+1,2n-i+1}(u)\xi = \left(\mu_i(-u-\rho) \pm \frac{\mu_i(u) - \mu_i(-u-\rho)}{2u+\rho}\right)\xi \quad \text{for} \quad 1 \le i \le n.$$

Recall from Section 1.2.2 that for an n-tuple $\mu \in \mathbb{C}^n$ we may define an irreducible highest weight representation of $U(\mathfrak{g}_{2n})$, which here will be denoted $M(\mu)$. Using the evaluation homomorphism $ev^\pm$, we can extend this to a $Y_\rho^\pm(\mathfrak{gl}_{2n})$-module with lowest weight satisfying

$$\mu_i(u) = 1 - \mu_i(u + (\rho \pm 1)/2)^{-1} \quad \text{for} \quad 1 \le i \le n. \tag{2.2.5}$$

Now, recall the tensor product of evaluation modules (2.1.1). The coproduct allows us to equip the space

$$M := L \otimes M(\mu) = L(\lambda^{(1)})_{c_1} \otimes L(\lambda^{(2)})_{c_2} \otimes \ldots \otimes L(\lambda^{(\ell)})_{c_\ell} \otimes M(\mu) \tag{2.2.6}$$

with the structure of a lowest weight $Y_\rho^\pm(\mathfrak{gl}_{2n})$-module. In particular, $S(u)$ acts on the space M by

$$S(u) \cdot M = \left(\prod_{i=1}^{\ell} \mathcal{L}_i(u - c_i)\right) \mathcal{L}^\pm(u) \left(\prod_{i=\ell}^{1} \mathcal{L}_i^t(-u-\rho-c_i)\right) M, , \tag{2.2.7}$$

where here we recall the convention of ordering products from left to right, so the leftmost operators in the above products are $\mathcal{L}_1(u - c_i)$ and $\mathcal{L}_\ell^t(-u-\rho-c_i)$ respectively, and we have introduced the "boundary" Lax operator

$$\mathcal{L}^\pm(u) := (id \otimes ev^\pm)(S(u)) = \sum_{i,j=1}^{2n} e_{ij} \otimes (\delta_{ij} - F_{ji}(u + (\rho \pm 1)/2)^{-1}). \tag{2.2.8}$$

This may be thought of as a generalisation of a K-matrix, in which case the chosen representation would be one-dimensional, but as of yet does not have a clear physical interpretation. Nevertheless, we include it here as a matter of mathematical interest.

Let $\xi \in M(\mu)$ be the lowest vector. Denote by η_i the lowest vector of $L(\lambda^{(i)})_{c_i}$ and set $\zeta = \eta_1 \otimes \ldots \otimes \eta_\ell \otimes \xi$. Then the submodule $Y_\rho^\pm(\mathfrak{gl}_{2n})\zeta$ of $Y_\rho^\pm(\mathfrak{gl}_{2n})$-module M is a lowest weight representation with a lowest vector ζ. It is given by

$$\lambda_i(u)\,\lambda_{2n-i+1}(-\rho-u)\,\mu_i(u) \quad \text{for} \quad 1 \le i \le n,$$

with $\lambda_i(u)$ defined in (2.1.2) and $\mu_i(u)$ defined in (2.2.5), see Proposition 4.2.11 in [Mo07]. To the best of our knowledge, there are currently no irreducibility criteria known for a tensor product of irreducible representations of $Y(\mathfrak{gl}_{2n})$ and $Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})$.

For the remainder of this chapter, we will study the problem of constructing eigenvectors of the transfer matrix defined by $\tau(u) = \mathrm{tr}_a\, S_a(u)$, for which we employ the nested algebraic Bethe ansatz.

2.2.2 Block decomposition

Just as with the $\mathfrak{gl}_n$ closed spin chain, or even the Heisenberg spin chain, the first step of the (nested) algebraic Bethe ansatz is the decomposition of the monodromy matrix in its auxiliary space into submatrices. In the $\mathfrak{gl}_n$ case, this took the form of (2.1.3), where each row was separated off one by one. For this spin chain, however, we employ a different strategy. Inspired by the arguments presented in [Rs91, DVK87], we decompose the monodromy matrix into equal $n \times n$ blocks. This is due to the redundancies contained within the generating matrix for a $\mathfrak{g}_{2n}$-symmetric system defined in RTT presentation.

We write matrices $T(u)$ and $S(u)$ in the block form:

$$T(u) = \begin{pmatrix} \overline{A}(u) & \overline{B}(u) \\ \overline{C}(u) & \overline{D}(u) \end{pmatrix}, \qquad S(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{2.2.9}$$

Our goal is to derive the algebraic relations between these smaller matrix operators (blocks), which we will make use of in the Bethe ansatz. We will denote the matrix elements of $A(u)$ by $a_{ij}(u)$ with $1 \le i,j \le n$, and similarly for matrices $B(u)$, $C(u)$ and $D(u)$, and their barred counterparts. Furthermore, we make use of barred indices $\bar{\imath} = n - i + 1$ for indices $1 \le i \le n$; note that in Section 1.2.2 this notation was used for indices running from 1 to $2n$.

Recall that $\mathbb{C}^{2n} \cong \mathbb{C}^2 \otimes \mathbb{C}^n$. Let e_{ij} with $1 \le i,j \le 2n$ denote the standard matrix units of $\mathrm{End}(\mathbb{C}^{2n})$. Moreover, let x_{ij} with $1 \le i,j \le 2$ (resp. e_{ij} with $1 \le i,j \le n$) denote the standard matrix units of $\mathrm{End}(\mathbb{C}^2)$ (resp. $\mathrm{End}(\mathbb{C}^n)$). Then, for any $1 \le i,j \le n$, we may write

$$\mathsf{e}_{ij} = x_{11} \otimes e_{ij}, \quad \mathsf{e}_{n+i,j} = x_{21} \otimes e_{ij}, \tag{2.2.10}$$

and similarly for $\mathsf{e}_{i,n+j}$ and $\mathsf{e}_{n+i,n+j}$. Hence any matrix $M \in \mathrm{End}(\mathbb{C}^{2n})$ with entries $(M)_{ij} \in \mathbb{C}$ can be equivalently written as

$$M = \sum_{a,b=1}^{2} x_{ab} \otimes [M]_{ab} \in \mathrm{End}(\mathbb{C}^2) \otimes \mathrm{End}(\mathbb{C}^n),$$

where $[M]_{ab} = \sum_{i,j=1}^{n} (M)_{i+n(a-1),j+n(b-1)}\, e_{ij}$ are blocks of M, viz. (2.2.9). Now let $M \in \mathrm{End}(\mathbb{C}^{2n} \otimes \mathbb{C}^{2n})$. Then we may write

$$M = \sum_{a,b,c,d=1}^{2} x_{ab} \otimes x_{cd} \otimes [M]_{abcd} \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n),$$

where $[M]_{abcd}$ are obtained as follows. Writing $M = \sum_{i,j,k,l=1}^{2n} (M)_{ijkl}\, \mathsf{e}_{ij} \otimes \mathsf{e}_{kl}$ we have

$$[M]_{abcd} = \sum_{i,j,k,l=1}^{n} (M)_{i+n(a-1),j+n(b-1),k+n(c-1),l+n(d-1)}\, e_{ij} \otimes e_{kl}. \tag{2.2.11}$$

Denote the R-matrix acting on $\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$ by $\mathbb{R}(u)$ and its t-transpose by $\mathbb{R}^t(u)$. Viewing them as elements in $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)[[u^{-1}]]$ and using (2.2.11) we recover the six-vertex block structure

$$\mathbb{R}(u) = \begin{pmatrix} R(u) & & & \\ & I & -u^{-1}P & \\ & -u^{-1}P & I & \\ & & & R(u) \end{pmatrix}, \quad \mathbb{R}^t(u) = \begin{pmatrix} I & & & \\ & R^t(u) & \mp u^{-1}Q & \\ & \mp u^{-1}Q & R^t(u) & \\ & & & I \end{pmatrix}, \tag{2.2.12}$$

where the operators inside the matrices are each acting on $\mathbb{C}^n \otimes \mathbb{C}^n$; note that $R^t(u) = I - u^{-1}Q$ and $Q = \sum_{1 \le i,j \le n} e_{ij} \otimes e_{\bar{j}\bar{\imath}}$ in both cases of $\mp$ above are of the orthogonal type, where here we have used the notation $\bar{\imath} = n - i + 1$.

In a similar way, the matrices $T_1(u) = T(u) \otimes I$ and $T_2(u) = I \otimes T(u)$, as elements of $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n) \otimes Y(\mathfrak{gl}_{2n})[[u^{-1}]]$, take the form

$$T_1(u) = \begin{pmatrix} \overline{A}_1(u) & & \overline{B}_1(u) & \\ & \overline{A}_1(u) & & \overline{B}_1(u) \\ \overline{C}_1(u) & & \overline{D}_1(u) & \\ & \overline{C}_1(u) & & \overline{D}_1(u) \end{pmatrix}, \quad T_2(u) = \begin{pmatrix} \overline{A}_2(u) & \overline{B}_2(u) & & \\ \overline{C}_2(u) & \overline{D}_2(u) & & \\ & & \overline{A}_2(u) & \overline{B}_2(u) \\ & & \overline{C}_2(u) & \overline{D}_2(u) \end{pmatrix}, \tag{2.2.13}$$

where $\overline{A}_1(u)$ means $\overline{A}(u) \otimes I \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n) \otimes Y(\mathfrak{g}_{2n})[[u^{-1}]]$ with I being the identity matrix, and similarly for the other blocks. Substituting (2.2.12) and (2.2.13) to the RTT relation allows us to rewrite the defining relations of $Y(\mathfrak{gl}_{2n})$ in terms of the matrices $\overline{A}(u)$, $\overline{B}(u)$, $\overline{C}(u)$ and $\overline{D}(u)$. The relations that we will need are:

$$R_{12}(u-v)\,\overline{A}_1(u)\,\overline{A}_2(v) = \overline{A}_2(v)\,\overline{A}_1(u)\,R_{12}(u-v), \tag{2.2.14}$$

$$R_{12}(u-v)\,\overline{B}_1(u)\,\overline{B}_2(v) = \overline{B}_2(v)\,\overline{B}_1(u)\,R_{12}(u-v), \tag{2.2.15}$$

$$R_{12}(u-v)\,\overline{D}_1(u)\,\overline{D}_2(v) = \overline{D}_2(v)\,\overline{D}_1(u)\,R_{12}(u-v), \tag{2.2.16}$$

$$\overline{C}_1(u)\,\overline{A}_2(v) = \overline{A}_2(v)\,\overline{C}_1(u)\,R_{12}(u-v) + \frac{P_{12}\,\overline{A}_1(u)\,\overline{C}_2(v)}{u-v}, \tag{2.2.17}$$

$$\overline{C}_1(u)\,\overline{D}_2(v) = R_{12}(v-u)\,\overline{D}_2(v)\,\overline{C}_1(u) - \frac{P_{12}\,\overline{D}_2(u)\,\overline{C}_1(v)}{u-v}, \tag{2.2.18}$$

$$\overline{D}_1(u)\,\overline{A}_2(v) - \overline{A}_2(v)\,\overline{D}_1(u) = \frac{P_{12}\,\overline{B}_1(u)\,\overline{C}_2(v) - \overline{B}_2(v)\,\overline{C}_1(u)\,P_{12}}{u-v}. \tag{2.2.19}$$

In particular, the coefficients of the matrix entries of $\overline{A}(u)$ generate a $Y(\mathfrak{gl}_n)$ subalgebra of $Y(\mathfrak{gl}_{2n})$. The same is true for $\overline{D}(u)$.

We now repeat the same steps for the twisted Yangian $Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})$. We substitute (2.2.12) to (2.2.2) and view matrices $S_1(u)$ and $S_2(u)$ as elements of $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n) \otimes Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})[[u^{-1}]]$, so that they take the same form as in (2.2.13). This allows us to write the defining relations of $Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})$ in terms of the matrices $A(u)$, $B(u)$, $C(u)$ and $D(u)$. The relations that we will need are:

$$\begin{aligned} A_2(v)B_1(u) &= R_{12}(u-v)B_1(u)R^t_{12}(-u-v-\rho)A_2(v) \\ &\qquad + \frac{P_{12}B_1(v)R^t_{12}(-u-v-\rho)A_2(u)}{u-v} \mp \frac{B_2(v)Q_{12}D_1(u)}{u+v+\rho}, \end{aligned} \tag{2.2.20}$$

$$\begin{aligned} &R_{12}(u-v)B_1(u)R^t_{12}(-u-v-\rho)B_2(v) \\ &\qquad = B_2(v)R^t_{12}(-u-v-\rho)B_1(u)R_{12}(u-v), \end{aligned} \tag{2.2.21}$$

$$\begin{aligned} &R_{12}(u-v)A_1(u)A_2(v) - A_2(v)A_1(u)R_{12}(u-v) \\ &\qquad = \mp\frac{R_{12}(u-v)B_1(u)Q_{12}C_2(v) - B_2(v)Q_{12}C_1(u)R_{12}(u-v)}{u+v+\rho}, \end{aligned} \tag{2.2.22}$$

$$\begin{aligned} C_1(u)A_2(v) &= A_2(v)R^t_{12}(-u-v-\rho)C_1(u)R_{12}(u-v) \\ &\qquad + \frac{P_{12}A_1(u)R^t_{12}(-u-v-\rho)C_2(v)}{u-v} \mp \frac{D_1(u)Q_{12}C_2(v)}{u+v+\rho}. \end{aligned} \tag{2.2.23}$$

It remains to cast the symmetry relation (2.2.3) in the block form. Observe that

$$S^t(u) = \begin{pmatrix} D^t(u) & \pm B^t(u) \\ \pm C^t(u) & A^t(u) \end{pmatrix}.$$

This allows us immediately to extract linear relations between matrices $A(u)$, $B(u)$, $C(u)$ and $D(u)$, of which we will need the following two only:

$$D^t(-u-\rho) = A(u) \pm \frac{1}{2u+\rho}\big(A(u) - A(-u-\rho)\big), \tag{2.2.24}$$

$$\pm B^t(-u-\rho) = B(u) \pm \frac{1}{2u+\rho}(B(u) - B(-u-\rho)). \tag{2.2.25}$$

Having defined the block relations, we now proceed to define the creation operators with which we will build an ansatz for the transfer matrix eigenvector. As with the $\mathfrak{gl}_n$ case, these will be taken from elements of the upper right "B-block". In the following section we define the creation operator, and derive relations between creation operators.

2.2.3 Creation operator for a single excitation

The key operators in the construction of the Bethe vector will come from the B block, viz. (2.2.9). However, rather than use a matrix of creation operators, we reinterpret $B(u)$ as a row vector in two *auxiliary spaces*, which will be denoted V_a and $V_{\tilde{a}}$, with components given by the matrix elements of $B(u)$.

Definition 2.2.3. *The creation operator is given by*

$$\beta(u) := \sum_{1\le i,j\le n} e_i^* \otimes e_j^* \otimes b_{\bar{\imath}j}(u) \in (\mathbb{C}^n)^* \otimes (\mathbb{C}^n)^* \otimes Y_\rho^\pm(\mathfrak{gl}_{2n})[[u^{-1}]], \tag{2.2.26}$$

where $\bar{\imath} = n - i + 1$.

The two auxiliary spaces in the above definition are labelled in the same order as the tensor product, that is, $\beta_{\tilde{a}a}(u) \in V_{\tilde{a}}^* \otimes V_a^* \otimes Y_\rho^\pm(\mathfrak{gl}_{2n})[[u^{-1}]]$. The exchange and symmetry relations involving the B operator may now be rewritten using the above notation. We introduce here the notation

$$p(u) = 1 \pm \frac{1}{2u+\rho}, \tag{2.2.27}$$

which will appear frequently in what follows.

Recall also the properties (1.2.22) and (1.2.23) of the matrix $Q := P^t$.

Lemma 2.2.4. *The creation operator satisfies the following identities:*

$$\begin{aligned}\beta_{\tilde{a}_1a_1}(u_1)\,\beta_{\tilde{a}_2a_2}(u_2)\,R_{a_1\tilde{a}_2}(-u_1-u_2-\rho)\,\check{R}_{\tilde{a}_1\tilde{a}_2}(u_1-u_2)&\\ = \beta_{\tilde{a}_1a_1}(u_2)\,\beta_{\tilde{a}_2a_2}(u_1)\,R_{a_1\tilde{a}_2}(-u_1-u_2-\rho)\,\check{R}_{a_1a_2}(u_1-u_2),&\end{aligned} \tag{2.2.28}$$

$$\beta_{\tilde{a}_ia_i}(u)\,Q_{a_ia} = \pm\left(p(-u-\rho)\,\beta_{\tilde{a}_ia_i}(-u-\rho) \pm \frac{\beta_{\tilde{a}_ia_i}(u)}{2u+\rho}\right)Q_{\tilde{a}_ia}\,Q_{a_ia}, \tag{2.2.29}$$

where $\check{R}(u) := PR(u)$.

Proof. We start by proving (2.2.28). From (2.2.21), begin by acting from the left with P_{12}, then use the defining property of the permutation operator to move it to the right on the r.h.s. of the equation to obtain

$$\check{R}_{12}(u_1-u_2)B_1(u_1)R^t_{12}(-u_1-u_2-\rho)B_2(u_2) = B_1(u_2)R^t_{12}(-u_1-u_2-\rho)B_2(u_1)\check{R}_{12}(u_1-u_2). \quad (2.2.30)$$

We want to rewrite this in terms of the creation operators defined in Definition 2.2.3. Choose bases for V_1 and V_2, then denote the matrix components of $R_{12}(-u_1-u_2-\rho)$ by $r_{i_1j_1i_2j_2}$, and the matrix components of $\check{R}_{12}(u_1-u_2)$ by $\check{r}_{i_1j_1i_2j_2}$. In components, (2.2.30) becomes

$$\sum_{j_1,j_2,k_1,k_2=1}^{n} \check{r}_{i_1j_1i_2j_2}\, b_{j_1k_1}(u_1)\, r_{k_1l_1\bar{k}_2\bar{j}_2}\, b_{k_2l_2}(u_2) = \sum_{j_1,j_2,k_1,k_2=1}^{n} b_{i_1j_1}(u_2)\, r_{j_1k_1\bar{j}_2\bar{\imath}_2}\, b_{j_2k_2}(u_1)\, \check{r}_{k_1l_1k_2l_2}.$$

Relabelling $i_1 \to \bar{\imath}_1$ and $i_2 \to \bar{\imath}_2$, and relabelling the summation indices $j_1 \to \bar{\jmath}_1$ and $j_2 \to \bar{\jmath}_2$ yields an equivalent expression:

$$\sum_{j_1,j_2,k_1,k_2=1}^{n} b_{\bar{\jmath}_1k_1}(u_1)\, b_{\bar{k}_2l_2}(u_2)\, r_{k_1l_1k_2j_2}\, \check{r}_{\bar{\imath}_1\bar{\jmath}_1\bar{\imath}_2\bar{\jmath}_2} = \sum_{j_1,j_2,k_1,k_2=1}^{n} b_{\bar{\imath}_1j_1}(u_2)\, b_{\bar{\jmath}_2k_2}(u_1)\, r_{j_1k_1j_2i_2}\, \check{r}_{k_1l_1k_2l_2}.$$

Finally, we note that $\check{r}_{\bar{\imath}_1\bar{\jmath}_1\bar{\imath}_2\bar{\jmath}_2} = \check{r}_{j_1i_1j_2i_2}$, as $\check{R}_{ab}(u)^{t_at_b} = \check{R}_{ab}(u)$. Then taking the tensor product with $e^*_{i_1} \otimes e^*_{l_1} \otimes e^*_{i_2} \otimes e^*_{l_2} \in V^*_{\tilde{a}_1} \otimes V^*_{a_1} \otimes V^*_{\tilde{a}_2} \otimes V^*_{a_2}$ and summing over these indices yields

$$\begin{aligned}&\beta_{\tilde{a}_1a_1}(u_1)\,\beta_{\tilde{a}_2a_2}(u_2)\,R_{a_1\tilde{a}_2}(-u_1-u_2-\rho)\,\check{R}_{\tilde{a}_1\tilde{a}_2}(u_1-u_2)\\ &\quad = \beta_{\tilde{a}_1a_1}(u_2)\,\beta_{\tilde{a}_2a_2}(u_1)\,R_{a_1\tilde{a}_2}(-u_1-u_2-\rho)\,\check{R}_{a_1a_2}(u_1-u_2),\end{aligned}$$

as required.

We now focus on (2.2.29). From (2.2.25) in matrix components, we make the assignment $u \mapsto -u-\rho$ and multiply by $\pm$ to obtain

$$b_{\bar{\jmath}\bar{\imath}}(u) = \pm p(-u-\rho)\, b_{ij}(-u-\rho) + \frac{b_{ij}(u)}{2u+\rho}.$$

Then, taking the tensor product with $e^*_j \otimes e^*_{\bar{\imath}} \in V^*_{\tilde{a}_i} \otimes V^*_{a_i}$, and summing over i, j yields the following expression in terms of the creation operator:

$$\beta_{\tilde{a}_ia_i}(u) = \pm p(-u-\rho)\beta_{a_i\tilde{a}_i}(-u-\rho) + \frac{\beta_{a_i\tilde{a}_i}(u)}{2u+\rho} = \pm\left(p(-u-\rho)\beta_{\tilde{a}_ia_i}(-u-\rho) \pm \frac{\beta_{\tilde{a}_ia_i}(u)}{2u+\rho}\right)P_{\tilde{a}_ia_i}.$$

To obtain (2.2.29) from here, we multiply on the right by the operator Q_{a_ia} and use the identity $P_{\tilde{a}_ia_i}Q_{a_ia} = Q_{\tilde{a}_ia}Q_{a_ia}$. □

2.2.4 Creation operator for multiple excitations

The next step is to generalize the creation operator $\beta(u)$ defined in (2.2.26) for multiple excitations. Choose $m \in \mathbb{N}$, the excitation number, and consider the tensor product space $W = V_{\tilde{a}_1} \otimes \cdots \otimes V_{\tilde{a}_m} \otimes V_{a_1} \otimes \cdots \otimes V_{a_m}$. Denote its dual by $W^* = V^*_{\tilde{a}_1} \otimes \cdots \otimes V^*_{\tilde{a}_m} \otimes V^*_{a_1} \otimes \cdots \otimes V^*_{a_m}$ and introduce an m-tuple of formal parameters $\boldsymbol{u} = (u_1, u_2, \ldots, u_m)$.

Definition 2.2.5. *The creation operator for m excitations is given in terms of the ordered product of β operators and R-matrices:*

$$\begin{aligned}\beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(\boldsymbol{u}) &= \prod_{i=1}^{m} \left(\beta_{\tilde{a}_i a_i}(u_i) \prod_{j=i-1}^{1} R_{a_j \tilde{a}_i}(-u_j - u_i - \rho) \right) \\ &\in W^* \otimes Y_\rho^{\pm}(\mathfrak{gl}_{2n})[u_1, \ldots, u_m][[u_1^{-1}, \ldots, u_m^{-1}]]. \end{aligned} \tag{2.2.31}$$

The insertion of R-matrices between the creation operators here differs from the $Y(\mathfrak{gl}_n)$ case; due to the fact that the B block matrices satisfy a twisted reflection equation-type relation (2.2.21), it is necessary to insert these R-matrices in order to exchange adjacent creation operators or their parameters. This may be compared to the equivalent block relation for $Y(\mathfrak{gl}_n)$ given by (2.2.15).

Note that the creation operator for m excitations satisfies the following recursive relation

$$\beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(\boldsymbol{u}) = \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_{m-1} a_{m-1}}(u_1, \ldots, u_{m-1})\, \beta_{\tilde{a}_m a_m}(u_m) \prod_{j=m-1}^{1} R_{a_j \tilde{a}_m}(-u_j - u_m - \rho). \tag{2.2.32}$$

Given $i \in \{1, \ldots, m-1\}$ denote by $\boldsymbol{u}_{i \leftrightarrow i+1}$ the m-tuple obtained from $\boldsymbol{u}$ by interchanging its i-th and $(i+1)$-th entries, namely

$$\boldsymbol{u}_{i \leftrightarrow i+1} = (u_1, u_2, \ldots, u_{i-1}, u_{i+1}, u_i, u_{i+2}, \ldots, u_m). \tag{2.2.33}$$

The Lemma below states a relation between the operators $\beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(\boldsymbol{u})$ and $\beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(\boldsymbol{u}_{i \leftrightarrow i+1})$ that will assist us in obtaining the explicit expressions of the so-called "unwanted terms" in Section 2.2.11.

Lemma 2.2.6. *The following identity holds:*

$$\beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(\boldsymbol{u}) = \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(\boldsymbol{u}_{i \leftrightarrow i+1})\, \check{R}_{a_i a_{i+1}}(u_i - u_{i+1})\, \check{R}^{-1}_{\tilde{a}_i \tilde{a}_{i+1}}(u_i - u_{i+1}) \tag{2.2.34}$$

for $1 \le i \le m-1$.

Proof. We use induction on m, with the basis case provided by (2.2.28). Assume the result holds for $m-1$ excitations. There are two cases to consider, depending on the spaces a_i, a_{i+1} on which $R_{a_i a_{i+1}}(u_i - u_{i+1})$ acts nontrivially. Consider first the case where $i < m-1$ and use the recursive relation (2.2.32):

$$
\begin{aligned}
\beta_{\tilde{a}_1 a_1 \dots \tilde{a}_m a_m}(\boldsymbol{u}) &= \beta_{\tilde{a}_1 a_1 \dots \tilde{a}_{m-1} a_{m-1}}(u_1, \dots, u_{m-1})\, \beta_{\tilde{a}_m a_m}(u_m) \prod_{j=m-1}^{1} R_{a_j \tilde{a}_m}(-u_j - u_m - \rho) \\
&= \beta_{\tilde{a}_1 a_1 \dots \tilde{a}_{m-1} a_{m-1}}(u_1, \dots, u_{i+1}, u_i, \dots, u_{m-1})\, \check{R}_{a_i a_{i+1}}(u_i - u_{i+1}) \\
&\quad \times \check{R}^{-1}_{\tilde{a}_i \tilde{a}_{i+1}}(u_i - u_{i+1})\, \beta_{\tilde{a}_m a_m}(u_m) \prod_{j=m-1}^{1} R_{a_j \tilde{a}_m}(-u_j - u_m - \rho).
\end{aligned}
$$

Notice that the matrix $\check{R}^{-1}_{\tilde{a}_i \tilde{a}_{i+1}}(u_i - u_{i+1})$ commutes with all matrices to the right of it, so it can be moved to the very right. The matrix $\check{R}_{a_i a_{i+1}}(u_i - u_{i+1})$ may be moved through the product of R-matrices using the (braided) Yang-Baxter equation:

$$
\begin{aligned}
&\check{R}_{a_i a_{i+1}}(u_i - u_{i+1}) R_{a_{i+1} \tilde{a}_m}(-u_{i+1} - u_m - \rho) R_{a_i \tilde{a}_m}(-u_i - u_m - \rho) \\
&\quad = R_{a_{i+1} \tilde{a}_m}(-u_i - u_m - \rho) R_{a_i \tilde{a}_m}(-u_{i+1} - u_m - \rho) \check{R}_{a_i a_{i+1}}(u_i - u_{i+1}).
\end{aligned}
$$

This then gives (2.2.34) for $i < m-1$. For $i = m-1$, we factorise the excitations as follows:

$$
\begin{aligned}
\beta_{\tilde{a}_1 a_1 \dots \tilde{a}_m a_m}(\boldsymbol{u}) &= \beta_{\tilde{a}_1 a_1 \dots \tilde{a}_{m-2} a_{m-2}}(u_1, \dots, u_{m-2})\, \beta_{\tilde{a}_{m-1} a_{m-1} \tilde{a}_m a_m}(u_{m-1}, u_m) \\
&\quad \times \prod_{j=m-2}^{1} \Big(R_{a_j \tilde{a}_{m-1}}(-u_j - u_{m-1} - \rho)\, R_{a_j \tilde{a}_m}(-u_j - u_m - \rho) \Big) \\
&= \beta_{\tilde{a}_1 a_1 \dots \tilde{a}_{m-2} a_{m-2}}(u_1, \dots, u_{m-2})\, \beta_{\tilde{a}_{m-1} a_{m-1} \tilde{a}_m a_m}(u_m, u_{m-1}) \\
&\quad \times \check{R}_{a_{m-1} a_m}(u_{m-1} - u_m)\, \check{R}^{-1}_{\tilde{a}_{m-1} \tilde{a}_m}(u_{m-1} - u_m) \\
&\quad \times \prod_{j=m-2}^{1} \Big(R_{a_j \tilde{a}_{m-1}}(-u_j - u_{m-1} - \rho)\, R_{a_j \tilde{a}_m}(-u_j - u_m - \rho) \Big).
\end{aligned}
$$

The matrix $\check{R}^{-1}_{\tilde{a}_{m-1} \tilde{a}_m}(u_{m-1} - u_m)$ may be moved through the product of R-matrices using another variant of the Yang-Baxter equation,

$$
\begin{aligned}
&\check{R}^{-1}_{\tilde{a}_{m-1} \tilde{a}_m}(u_{m-1} - u_m)\, R_{a_j \tilde{a}_{m-1}}(-u_j - u_{m-1} - \rho)\, R_{a_j \tilde{a}_m}(-u_j - u_m - \rho) \\
&\quad = R_{a_j \tilde{a}_{m-1}}(-u_j - u_m - \rho)\, R_{a_j \tilde{a}_m}(-u_j - u_{m-1} - \rho)\, \check{R}^{-1}_{\tilde{a}_{m-1} \tilde{a}_m}(u_{m-1} - u_m).
\end{aligned}
$$

Then, rearranging the commuting matrices in the expression, we reconstruct the full excitation vector and arrive at (2.2.34) for $i = m-1$. This completes the induction. □

2.2.5 Rewriting the AB exchange relation

We now consider the exchange relation between the $A(u)$ operator and the creation operator (2.2.26). Looking at the relation (2.2.20), we see that it has two "unwanted terms", one of which includes the $D(u)$ operator. This mixing appears to cause a problem for the definition of a nested monodromy matrix, and we endeavour to rewrite this relation in a more useful form.

Lemma 2.2.7. *The following identity holds:*

$$\begin{aligned} A_a(v)\,\beta_{\tilde{a}_i a_i}(u) &= \beta_{\tilde{a}_i a_i}(u)\,R^t_{\tilde{a}_i a}(u-v)\,R^t_{a_i a}(-u-v-\rho)\,A_a(v) \\ &\quad + \frac{\beta_{\tilde{a}_i a_i}(v)}{u-v}\,Q_{\tilde{a}_i a}\,R^t_{a_i a}(-2u-\rho)\,A_a(u) \\ &\quad \mp \frac{p(-u-\rho)}{u+v+\rho}\,\beta_{\tilde{a}_i a_i}(v)\,Q_{\tilde{a}_i a}\,Q_{a_i a}\,A_a(-u-\rho). \end{aligned} \tag{2.2.35}$$

Proof. We introduce the following rule for obtaining expressions in terms of the β operator from those in terms of B operator. Let $X_{\tilde{a}} \in \mathrm{End}(V_{\tilde{a}})$ and $Y_a \in \mathrm{End}(V_a)$. Considering the components of $\beta_{\tilde{a}a}(u)\,X^t_{\tilde{a}}\,Y_a$, we have

$$\begin{aligned} \beta_{\tilde{a}a}(u)\,X^t_{\tilde{a}}\,Y_a &= \sum_{1\le i,j,k,l,r,s\le n} (e^*_k \otimes e^*_l \otimes b_{\bar{k}l}(u))(e_{ri} \otimes e_{sj} \otimes x_{\bar{\imath}\bar{r}}\,y_{sj}) \\ &= \sum_{1\le i,j,k,l\le n} e_i \otimes e_j \otimes b_{\bar{k}l}(u)\,x_{\bar{\imath}\bar{k}}\,y_{lj}, \end{aligned}$$

so that $(\beta_{\tilde{a}a}(u)\,X^t_{\tilde{a}}\,Y_a)_{\bar{\imath}j} = \sum_{1\le k,l\le n} b_{kl}(u)\,x_{ik}\,y_{lj}$. On the other hand, taking the (i,j)-th matrix element of the expression $X_a\,B_a(u)\,Y_a$ for any $X_a, Y_a \in \mathrm{End}(V_a)$ we obtain

$$(X_a\,B_a(u)\,Y_a)_{ij} = \sum_{1\le k,l\le n} x_{ik}\,b_{kl}(u)\,y_{lj} = (\beta_{\tilde{a}a}(u)\,X^t_{\tilde{a}}\,Y_a)_{\bar{\imath}j}. \tag{2.2.36}$$

Using this rule, we can rewrite (2.2.20) as

$$\begin{aligned} A_a(v)\beta_{\tilde{a}_i a_i}(u) &= \beta_{\tilde{a}_i a_i}(u)R^t_{\tilde{a}_i a}(u-v)R^t_{a_i a}(-u-v-\rho)A_a(v) \\ &\quad + \frac{\beta_{\tilde{a}_i a_i}(v)}{u-v}Q_{\tilde{a}_i a}R^t_{a_i a}(-u-v-\rho)A_a(u) \\ &\quad \mp \frac{\beta_{\tilde{a}_i a_i}(v)}{u+v+\rho}Q_{\tilde{a}_i a}Q_{a_i a}D^t_a(u), \end{aligned}$$

where the identities $X_1 = P_{12}X_2P_{12}$ and $Q_{12}X_1 = Q_{12}X^t_2$ have been used. From here, the symmetry relation (2.2.24) may be used to obtain

$$\begin{aligned} A_a(v)\beta_{\tilde{a}_i a_i}(u) &= \beta_{\tilde{a}_i a_i}(u)R^t_{\tilde{a}_i a}(u-v)R^t_{a_i a}(-u-v-\rho)A_a(v) \\ &\quad + \frac{\beta_{\tilde{a}_i a_i}(v)}{u-v}Q_{\tilde{a}_i a}R^t_{a_i a}(-u-v-\rho)A_a(u) \\ &\quad \mp \frac{\beta_{\tilde{a}_i a_i}(v)}{u+v+\rho}Q_{\tilde{a}_i a}Q_{a_i a}\left(p(-u-\rho)\,A_a(-u-\rho) \pm \frac{A_a(u)}{2u+\rho}\right). \end{aligned}$$

We note that

$$\frac{R^t_{a_i a}(-u-v-\rho)}{u-v} - \frac{Q_{a_i a}}{(2u+\rho)(u+v+\rho)} = \frac{1}{u-v}\left(I + \left(1 - \frac{u-v}{2u+\rho}\right)\frac{Q_{a_i a}}{u+v+\rho}\right) = \frac{R^t(-2u-\rho)}{u-v}.$$

Following these manipulations, we obtain

$$\begin{aligned} A_a(v)\,\beta_{\tilde{a}_i a_i}(u) &= \beta_{\tilde{a}_i a_i}(u)\,R^t_{\tilde{a}_i a}(u-v)\,R^t_{a_i a}(-u-v-\rho)\,A_a(v) \\ &+ \frac{\beta_{\tilde{a}_i a_i}(v)}{u-v}\,Q_{\tilde{a}_i a}\,R^t_{a_i a}(-2u-\rho)\,A_a(u) \\ &\mp \frac{p(-u-\rho)}{u+v+\rho}\,\beta_{\tilde{a}_i a_i}(v)\,Q_{\tilde{a}_i a}\,Q_{a_i a}\,A_a(-u-\rho), \end{aligned} \tag{2.2.37}$$

as required. □

This relation (2.2.35) is convenient as it does not feature the D operator, so the relation can be used repeatedly in the presence of multiple creation operators. However, to obtain the most convenient form of (2.2.35), we must consider the action of $p(v)A_a(v) + p(-v-\rho)A_a(-v-\rho)$ on $\beta_{\tilde{a}_i a_i}(u)$ rather than of $A_a(v)$ alone (the motivation for this construction will be explained in Section 2.2.8). Introduce the following notation for a symmetrised combination of functions or operators,

$$\{f(v)\}^v := f(v) + f(-v-\rho). \tag{2.2.38}$$

Lemma 2.2.8. *The following identity holds:*

$$\begin{aligned} &\{p(v)\,A_a(v)\}^v\,\beta_{\tilde{a}_i a_i}(u) \qquad (2.2.39)\\ &= \beta_{\tilde{a}_i a_i}(u)\,\left\{p(v)\,R^t_{\tilde{a}_i a}(u-v)\,R^t_{a_i a}(-u-v-\rho)\,A_a(v)\right\}^v \\ &+ \frac{1}{p(u)}\left\{\frac{p(v)}{u-v}\beta_{\tilde{a}_i a_i}(v)\right\}^v \operatorname*{Res}_{w\to u}\left[\left\{p(w)\,R^t_{\tilde{a}_i a}(u-w)\,R^t_{a_i a}(-u-w-\rho)\,A_a(w)\right\}^w\right]. \end{aligned} \tag{2.2.40}$$

Proof. Starting from (2.2.35), multiplying by $p(v)$ and symmetrising using (2.2.38), we obtain

$$\begin{aligned} \{p(v)\,A_a(v)\}^v\,\beta_{\tilde{a}_i a_i}(u) &= \beta_{\tilde{a}_i a_i}(u)\,\left\{R^t_{\tilde{a}_i a}(u-v)\,R^t_{a_i a}(-u-v-\rho)\,p(v)\,A_a(v)\right\}^v \\ &+ \left\{\frac{p(v)}{u-v}\,\beta_{\tilde{a}_i a_i}(v)\right\}^v Q_{\tilde{a}_i a}\,R^t_{a_i a}(-2u-\rho)\,A_a(u) \\ &\mp p(-u-\rho)\left\{\frac{p(v)}{u+v+\rho}\,\beta_{\tilde{a}_i a_i}(v)\right\}^v Q_{\tilde{a}_i a}\,Q_{a_i a}\,A_a(-u-\rho). \end{aligned} \tag{2.2.41}$$

We will show that this is equivalent to (2.2.39) term by term, separating the terms by the parameter carried by $A_a(\cdot)$. Note that the term containing $A_a(v)$ is already the same in both (2.2.39) and (2.2.41). For the remaining terms, containing $A_a(u)$ and $A_a(-u-\rho)$, we will work backwards from

(2.2.39). Let

$$U = \frac{1}{p(u)} \left\{ \frac{p(v)}{u-v} \beta_{\tilde{a}_i a_i}(v) \right\}^v \operatorname*{Res}_{w \to u} \left[\left\{ p(w) \, R^t_{\tilde{a}_i a}(u-w) \, R^t_{a_i a}(-u-w-\rho) \, A_a(w) \right\}^w \right].$$

Furthermore, expand the symmetriser inside the residue so that $U = U_+ + U_-$, where

$$\begin{aligned} U_+ &= \frac{1}{p(u)} \left\{ \frac{p(v)}{u-v} \beta_{\tilde{a}_i a_i}(v) \right\}^v \operatorname*{Res}_{w \to u} \Big[p(w) \, R^t_{\tilde{a}_i a}(u-w) \, R^t_{a_i a}(-u-w-\rho) \, A_a(w) \Big], \\ U_- &= \frac{1}{p(u)} \left\{ \frac{p(v)}{u-v} \beta_{\tilde{a}_i a_i}(v) \right\}^v \operatorname*{Res}_{w \to u} \Big[p(-w-\rho) \, R^t_{\tilde{a}_i a}(u+w+\rho) \, R^t_{a_i a}(w-u) \, A_a(-w-\rho) \Big]. \end{aligned}$$

Focussing first on U_+, we evaluate the residue to obtain

$$U_+ = \left\{ \frac{p(v)}{u-v} \beta_{\tilde{a}_i a_i}(v) \right\}^v Q_{\tilde{a}_i a} \, R^t_{a_i a}(-2u-\rho) \, A_a(u).$$

This now matches the term containing $A_a(u)$ in (2.2.41). It remains to show that U_- is equal to the term containing $A_a(-u-\rho)$ in (2.2.41). Again evaluating the residue, we obtain

$$\begin{aligned} U_- &= -\left\{ \frac{p(v)}{u-v} \beta_{\tilde{a}_i a_i}(v) \right\}^v \frac{p(-u-\rho)}{p(u)} \, R^t_{\tilde{a}_i a}(2u+\rho) \, Q_{a_i a} \, A_a(-u-\rho) \\ &= -\left\{ \frac{p(v)}{u-v} \left(\beta_{\tilde{a}_i a_i}(v) \, Q_{a_i a} - \beta_{\tilde{a}_i a_i}(v) \, \frac{Q_{\tilde{a}_i a} Q_{a_i a}}{2u+\rho} \right) \right\}^v \frac{p(-u-\rho)}{p(u)} \, A_a(-u-\rho). \end{aligned}$$

We now apply the symmetry relation (2.2.29), so

$$\begin{aligned} U_- = -\Bigg\{ \frac{p(v)}{u-v} \bigg(\pm \, p(-v-\rho) \, \beta_{\tilde{a}_i a_i}(-v-\rho) + \frac{\beta_{\tilde{a}_i a_i}(v)}{2v+\rho} - \frac{\beta_{\tilde{a}_i a_i}(v)}{2u+\rho} \bigg) \Bigg\}^v \\ \times \frac{p(-u-\rho)}{p(u)} \, Q_{\tilde{a}_1 a} \, Q_{a_i a} \, A_a(-u-\rho). \end{aligned}$$

Since it lies within the symmetriser, the term containing $\beta_{\tilde{a}_i a_i}(-v-\rho)$ can be rewritten in terms of $\beta_{\tilde{a}_i a_i}(v)$ to obtain

$$\begin{aligned} U_- = -\Bigg\{ \bigg(\pm \frac{p(-v-\rho)}{u+v+\rho} + \frac{1}{u-v} \bigg(\frac{1}{2v+\rho} - \frac{1}{2u+\rho} \bigg) \bigg) p(v) \, \beta_{\tilde{a}_i a_i}(v) \Bigg\}^v \\ \times \frac{p(-u-\rho)}{p(u)} \, Q_{\tilde{a}_i a} \, Q_{a_i a} \, A_a(-u-\rho). \end{aligned}$$

All that remains are algebraic manipulations:

$$\begin{aligned}
U_- &= -\left\{\left(\pm\frac{p(-v-\rho)}{u+v+\rho}+\frac{2}{(2v+\rho)(2u+\rho)}\right)p(v)\,\beta_{\tilde{a}_ia_i}(v)\right\}^v\frac{p(-u-\rho)}{p(u)}Q_{\tilde{a}_1a}Q_{a_ia}A_a(-u-\rho)\\
&= -\left\{\left(\pm 1-\frac{1}{2v+\rho}+\frac{2u+2v+2\rho}{(2v+\rho)(2u+\rho)}\right)\frac{p(v)}{u+v+\rho}\beta_{\tilde{a}_ia_i}(v)\right\}^v\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times\frac{p(-u-\rho)}{p(u)}Q_{\tilde{a}_1a}Q_{a_ia}A_a(-u-\rho)\\
&= -\left\{\left(\pm 1+\frac{1}{2u+\rho}\right)\frac{p(v)}{u+v+\rho}\beta_{\tilde{a}_ia_i}(v)\right\}^v\frac{p(-u-\rho)}{p(u)}Q_{\tilde{a}_1a}Q_{a_ia}A_a(-u-\rho)\\
&= \mp p(-u-\rho)\left\{\frac{p(v)}{u+v+\rho}\beta_{\tilde{a}_ia_i}(v)\right\}^v Q_{\tilde{a}_1a}Q_{a_ia}A_a(-u-\rho).
\end{aligned}$$

This matches the term containing $A_a(-u-\rho)$ in (2.2.41) and completes the proof. □

2.2.6 The AB exchange relation for multiple excitations

We want to move $\{p(v)\,A_a(v)\}^v$ through the operator $\beta_{\tilde{a}_1a_1\ldots\tilde{a}_ma_m}(\boldsymbol{u})$. Each time $\{p(v)\,A_a(v)\}^v$ is moved through one of the excitations $\beta_{\tilde{a}_ia_i}(u_i)$ using (2.2.35), we obtain a term, where the parameter v of $\{p(v)\,A_a(v)\}^v$ is unchanged. We will call this term the *wanted term*. All the additional terms will be called the *unwanted terms*; we will denote them by UWT and consider their exact form in Section 2.2.11. Focussing on the wanted term at each step, $\{p(v)\,A_a(v)\}^v$ accrues R-matrices as it moves through the excitations. In the following Lemma, we will show that these R-matrices may be moved through those appearing in the operator $\beta_{\tilde{a}_1a_1\ldots\tilde{a}_ma_m}(\boldsymbol{u})$.

Lemma 2.2.9. *The following exchange relation holds*

$$\begin{aligned}
&\left(\prod_{k=1}^{i-1}R^t_{\tilde{a}_ka}(u_k-v)\right)\left(\prod_{l=1}^{i-1}R^t_{a_la}(-u_l-v-\rho)\right)A_a(v)\,\beta_{\tilde{a}_ia_i}(u_i)\prod_{j=i-1}^{1}R_{a_j\tilde{a}_i}(-u_j-u_i-\rho)\\
&\quad=\beta_{\tilde{a}_ia_i}(u_i)\left(\prod_{j=i-1}^{1}R_{a_j\tilde{a}_i}(-u_j-u_i-\rho)\right)\left(\prod_{k=1}^{i}R^t_{\tilde{a}_ka}(u_k-v)\right)\left(\prod_{l=1}^{i}R^t_{a_la}(-u_l-v-\rho)\right)A_a(v)\\
&\qquad\quad+UWT.
\end{aligned}$$

Proof. We begin by using (2.2.35) and focus on the wanted terms only:

$$\begin{aligned}
&\left(\prod_{k=1}^{i-1}R^t_{\tilde{a}_ka}(u_k-v)\right)\left(\prod_{l=1}^{i-1}R^t_{a_la}(-u_l-v-\rho)\right)A_a(v)\,\beta_{\tilde{a}_ia_i}(u_i)\prod_{j=i-1}^{1}R_{a_j\tilde{a}_i}(-u_j-u_i-\rho)\\
&\quad=\left(\prod_{k=1}^{i-1}R^t_{\tilde{a}_ka}(u_k-v)\right)\left(\prod_{l=1}^{i-1}R^t_{a_la}(-u_l-v-\rho)\right)\beta_{\tilde{a}_ia_i}(u_i)\,R^t_{\tilde{a}_ia}(u_i-v)\\
&\qquad\times R^t_{a_ia}(u_i-v)A_a(v)\prod_{j=i-1}^{1}R_{a_j\tilde{a}_i}(-u_j-u_i-\rho)+UWT
\end{aligned}$$

yielding

$$\beta_{\tilde{a}_i a_i}(u_i)\left(\prod_{k=1}^{i-1} R^t_{\tilde{a}_k a}(u_k - v)\right)\left(\prod_{l=1}^{i-1} R^t_{a_l a}(-u_l - v - \rho)\right) R^t_{\tilde{a}_i a}(u_i - v)$$
$$\times \left(\prod_{j=i-1}^{1} R_{a_j \tilde{a}_i}(-u_j - u_i - \rho)\right) R^t_{a_i a}(u_i - v) A_a(v) + UWT.$$

All that remains is to rearrange the product of R-matrices in the centre of the expression. The matrices can be reordered using the Yang-Baxter equation

$$\begin{aligned} &R^t_{a_{i-1}a}(-u_{i-1} - v - \rho)\, R^t_{\tilde{a}_i a}(u_i - v)\, R_{a_{i-1}\tilde{a}_i}(-u_{i-1} - u_i - \rho) \\ &\quad = R_{a_{i-1}\tilde{a}_i}(-u_{i-1} - u_i - \rho)\, R^t_{\tilde{a}_i a}(u_i - v)\, R^t_{a_{i-1}a}(-u_{i-1} - v - \rho). \end{aligned}$$

Thus the product of R-matrices becomes

$$\begin{aligned} &\left(\prod_{l=1}^{i-1} R^t_{a_l a}(-u_l - v - \rho)\right) R^t_{\tilde{a}_i a}(u_i - v)\left(\prod_{j=i-1}^{1} R_{a_j \tilde{a}_i}(-u_j - u_i - \rho)\right) \\ &= \left(\prod_{l=1}^{i-2} R^t_{a_l a}(-u_l - v - \rho)\right) R_{a_{i-1}\tilde{a}_i}(-u_{i-1} - u_i - \rho) R^t_{\tilde{a}_i a}(u_i - v) \\ &\quad \times R^t_{a_{i-1}a}(-u_{i-1} - v - \rho)\left(\prod_{j=i-2}^{1} R_{a_j \tilde{a}_i}(-u_j - u_i - \rho)\right) \\ &= R_{a_{i-1}\tilde{a}_i}(-u_{i-1} - u_i - \rho)\left(\prod_{l=1}^{i-2} R^t_{a_l a}(-u_l - v - \rho)\right) R^t_{\tilde{a}_i a}(u_i - v) \\ &\quad \times \left(\prod_{j=i-2}^{1} R_{a_j \tilde{a}_i}(-u_j - u_i - \rho)\right) R^t_{a_{i-1}a}(-u_{i-1} - v - \rho) \\ &= \left(\prod_{j=i-1}^{1} R_{a_j \tilde{a}_i}(-u_j - u_i - \rho)\right) R^t_{\tilde{a}_i a}(u_i - v)\left(\prod_{l=1}^{i-1} R^t_{a_l a}(-u_l - v - \rho)\right), \end{aligned}$$

where the last equality is achieved by inductively applying the same argument. Putting this together, and noting that the R^t-matrices all commute with the R-matrices, we arrive to

$$\begin{aligned} &\left(\prod_{k=1}^{i-1} R^t_{\tilde{a}_k a}(u_k - v)\right)\left(\prod_{l=1}^{i-1} R^t_{a_l a}(-u_l - v - \rho)\right) A_a(v)\, \beta_{\tilde{a}_i a_i}(u_i) \prod_{j=i-1}^{1} R_{a_j \tilde{a}_i}(-u_j - u_i - \rho) \\ &\quad = \beta_{\tilde{a}_i a_i}(u_i)\left(\prod_{j=i-1}^{1} R_{a_j \tilde{a}_i}(-u_j - u_i - \rho)\right)\left(\prod_{k=1}^{i} R^t_{\tilde{a}_k a}(u_k - v)\right)\left(\prod_{l=1}^{i} R^t_{a_l a}(-u_l - v - \rho)\right) A_a(v) \\ &\qquad + UWT \end{aligned}$$

as required. □

Applying this result to the product of m such excitations in (2.2.31) yields

$$A_a(v)\,\beta_{\tilde a_1 a_1\dots\tilde a_m a_m}(\boldsymbol{u}) = \beta_{\tilde a_1 a_1\dots\tilde a_m a_m}(\boldsymbol{u})\left(\prod_{k=1}^{m} R^t_{\tilde a_k a}(u_k - v)\right)\left(\prod_{l=1}^{m} R^t_{a_l a}(-u_l - v - \rho)\right)A_a(v) + UWT.$$

We define the matrix on the right side to be the *nested monodromy matrix*,

$$T_a(v;\boldsymbol{u}) := \left(\prod_{k=1}^{m} R^t_{\tilde a_k a}(u_k - v)\right)\left(\prod_{l=1}^{m} R^t_{a_l a}(-u_l - v - \rho)\right)A_a(v). \tag{2.2.42}$$

Its matrix entries will be denoted by $t_{ij}(v;\boldsymbol{u})$. The matrix $T_a(v;\boldsymbol{u})$ allows us to write the above identity more compactly,

$$A_a(v)\,\beta_{\tilde a_1 a_1\dots\tilde a_m a_m}(\boldsymbol{u}) = \beta_{\tilde a_1 a_1\dots\tilde a_m a_m}(\boldsymbol{u})\,T_a(v;\boldsymbol{u}) + UWT,$$

which leads to the following result.

Corollary 2.2.10. *The AB exchange relation for the creation operator of multiple excitations has the form*

$$\{p(v)\,A_a(v)\}^v\,\beta_{\tilde a_1 a_1\dots\tilde a_m a_m}(\boldsymbol{u}) = \beta_{\tilde a_1 a_1\dots\tilde a_m a_m}(\boldsymbol{u})\,\{p(v)\,T_a(v;\boldsymbol{u})\}^v + UWT. \qquad \square$$

2.2.7 Exchange relations for the nested monodromy matrix

In this section we introduce a vector space M', called the *nested vacuum sector*, on which the nested monodromy matrix $T(v;\boldsymbol{u})$ satisfies the usual RTT relation. This allows us to identify $T(v;\boldsymbol{u})$ as the monodromy matrix for a residual $Y(\mathfrak{gl}_n)$-system. The space M' is then interpreted as the full quantum space of this residual system.

We start by introducing certain subspaces of the evaluation modules $M(\mu)$ and $L(\lambda^{(i)})_{c_i}$ that will be building blocks of the space M'. Denote by $M^0(\mu)$ the subspace of the evaluation module $M(\mu)$ of the twisted Yangian $Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})$ consisting of vectors annihilated by the operator $C(u)$ of the matrix $S(u)$, namely

$$M^0(\mu) := \{\zeta \in M(\mu) : c_{ij}(u)\zeta = 0 \ \text{ for } \ 1 \le i,j \le n\}.$$

The subspace $M^0(\mu)$ corresponds to the natural embedding $\mathfrak{gl}_n \subset \mathfrak{g}_{2n}$ with $\mathfrak{g}_{2n} = \mathfrak{so}_{2n}$ or $\mathfrak{sp}_{2n}$ (generated by F_{ij} with $1 \le i,j \le n$, *viz.* (1.2.18-1.2.19)) and is an irreducible $\mathfrak{gl}_n$-representation with highest weight $\mu = (\mu_1,\dots,\mu_n)$. The space $M^0(\mu)$ is stable under the action of the operator $A(u)$ of the matrix $S(u)$. Moreover, $A(u)$ satisfies the usual RTT relation on this space. Indeed, applying equality (2.2.23) to $M^0(\mu)$ yields $C_1(v)\,A_2(u)\,M^0(\mu) = 0$. Applying (2.2.22) instead we obtain

$$R(u-v)\,A_1(u)\,A_2(v)\,\zeta = A_2(v)\,A_1(u)\,R(u-v)\,\zeta$$

for all $\zeta \in M^0(\mu)$. We thus have the following.

Lemma 2.2.11. *The mapping*

$$Y(\mathfrak{gl}_n) \to Y_\rho^\pm(\mathfrak{gl}_{2n}), \quad T(u) \mapsto A(u)$$

equips the space $M^0(\mu)$ *with a structure of a lowest weight* $Y(\mathfrak{gl}_n)$*-module with the lowest weight given by* (2.2.5). $\square$

Note that the operator $A(u)$ of the matrix $S(u)$ acts on the space $M^0(\mu)$ via the Lax operator

$$\mathcal{L}^{\pm,0}(u) := \sum_{i,j=1}^{n} e_{ij} \otimes (\delta_{ij} - F_{ji}(u + (\rho \pm 1)/2)^{-1}), \tag{2.2.43}$$

which is the restriction of $\mathcal{L}^\pm(u)$ defined in (2.2.8) to the operator $A(u)$.

Next, we denote by $L^0(\lambda^{(k)})_{c_k}$ the subspace of the evaluation module $L(\lambda^{(k)})_{c_k}$ of $Y(\mathfrak{gl}_{2n})$ consisting of vectors annihilated by the operator $\overline{C}(u)$ of the matrix $T(u)$, namely

$$L^0(\lambda^{(k)})_{c_k} := \{\zeta \in L(\lambda^{(k)})_{c_k} \,:\, \bar{c}_{ij}(u)\zeta = 0 \ \text{ for } \ 1 \le i,j \le n\}. \tag{2.2.44}$$

The subspace $L^0(\lambda^{(k)})_{c_k}$ corresponds to the natural embedding $\mathfrak{gl}_n \oplus \mathfrak{gl}_n \subset \mathfrak{gl}_{2n}$ (generated by E_{ij} with $1 \le i,j \le n$ and $n < i,j \le 2n$) and is isomorphic to a tensor product of irreducible $\mathfrak{gl}_n$-representations $L(\lambda'^{(k)}) \otimes L(\lambda''^{(k)})$ with the highest weights $\lambda'^{(k)} = (\lambda_1^{(k)}, \ldots, \lambda_n^{(k)})$ and $\lambda''^{(k)} = (\lambda_{n+1}^{(k)}, \ldots, \lambda_{2n}^{(k)})$. Indeed, applying equality (2.2.17) to $L^0(\lambda^{(k)})_{c_k}$ yields $\overline{C}_1(u)\overline{A}_2(v)\, L^0(\lambda^{(k)})_{c_k} = 0$. Applying (2.2.18) instead we obtain $\overline{C}_1(u)\overline{D}_2(v)\, L^0(\lambda^{(k)})_{c_k} = 0$. Moreover, applying (2.2.19) to $L^0(\lambda^{(i)})_{c_i}$ we get $[\overline{D}_1(u), \overline{A}_2(v)]\, L^0(\lambda^{(k)})_{c_k} = 0$. This, together with (2.2.14) and (2.2.16), implies the following.

Lemma 2.2.12. *Each of the mappings*

$$Y(\mathfrak{gl}_n) \to Y(\mathfrak{gl}_{2n}), \quad T(u) \mapsto \overline{A}(u) \quad \text{and} \quad T(u) \mapsto \overline{D}(u)$$

is a homomorphism of algebras. Moreover, these mappings equip the spaces $L(\lambda'^{(k)})$ *and* $L(\lambda''^{(k)})$ *with a structure of a lowest weight* $Y(\mathfrak{gl}_n)$*-module with the lowest weight given by* $\lambda_k'^{(k)}(u) = \lambda_k^{(k)}(u)$ *and* $\lambda_k''^{(k)}(u) = \lambda_{n+k}^{(k)}(u)$*, respectively, for* $1 \le k \le n$. $\square$

Denote the corresponding $Y(\mathfrak{gl}_n)$-modules by $L^0(\lambda'^{(k)})_{c_k}$ and $L^0(\lambda''^{(k)})_{c_k}$, respectively. The operator $\overline{A}(u)$ of the matrix $T(u)$ of $Y(\mathfrak{gl}_{2n})$ acts on the space $L^0(\lambda''^{(k)})_{c_k}$ as the identity operator, and on the space $L^0(\lambda'^{(k)})_{c_k}$ via the restriction of the Lax operator ,

$$\mathcal{L}^0(u - c_k) := \sum_{i,j=1}^{n} e_{ij} \otimes (\delta_{ij} - E_{ji}(u - c_k)^{-1}). \tag{2.2.45}$$

Likewise, the operator $\overline{D}^t(u)$ of the transposed matrix $T^t(u)$ acts on the space $L^0(\lambda'^{(k)})_{c_k}$ as the identity operator, and on the space $L^0(\lambda''^{(k)})_{c_k}$ via the transposed Lax operator $(\mathcal{L}^0(u-c_k))^t$.

We are now ready to introduce the *vacuum sector* $M^0 \subset M$ by

$$M^0 := L^0(\lambda^{(1)})_{c_1} \otimes \cdots \otimes L^0(\lambda^{(\ell)})_{c_\ell} \otimes M^0(\mu). \tag{2.2.46}$$

Lemma 2.2.13. *The space M^0 is stable under the action of the operator $A(u)$ of the matrix $S(u)$.*

Proof. We start by showing that operator $C(u)$ of the matrix $S(u)$ annihilates the space M^0: $c_{ij}(v) \cdot M^0 = 0$. We use induction on ℓ. For $\ell = 0$ we have $M^0 = M^0(\mu)$ and $c_{ij}(v) M^0(\mu) = 0$, by definition (2.2.46). For any $\ell \geq 1$ denote $M^{(\ell-1)} := L^0(\lambda^{(1)})_{c_1} \otimes \cdots \otimes L^0(\lambda^{(\ell-1)})_{c_{\ell-1}} \otimes M^0(\mu)$. Let $\zeta \in M^{(\ell-1)}$ and $\zeta' \in L^0(\lambda^{(\ell)})_{c_\ell}$ be any nonzero vectors. Using (2.2.4) and the notation (2.2.9) we find

$$\begin{aligned} c_{ij}(u) \cdot (\zeta' \otimes \zeta) &= \sum_{k,l=1}^{n} \Big(\bar{c}_{ik}(u) \bar{d}_{\bar{\jmath}\bar{l}}(-u-\rho)\zeta' \otimes a_{kl}(u) \cdot \zeta \pm \bar{c}_{ik}(u)\bar{c}_{\bar{\jmath}\bar{l}}(-u-\rho)\zeta' \otimes b_{kl}(u) \cdot \zeta \\ &\qquad + \bar{d}_{ik}(u)\bar{d}_{\bar{\jmath}\bar{l}}(-u-\rho)\zeta' \otimes c_{kl}(u) \cdot \zeta \pm \bar{d}_{ik}(u)\bar{c}_{\bar{\jmath}\bar{l}}(-u-\rho)\zeta' \otimes d_{kl}(u) \cdot \zeta \Big) \\ &= \sum_{k,l=1}^{n} \Big(\bar{c}_{ik}(u)\bar{d}_{\bar{\jmath}\bar{l}}(-u-\rho)\zeta' \otimes a_{kl}(u) \cdot \zeta + \bar{d}_{ik}(u)\bar{d}_{\bar{\jmath}\bar{l}}(-u-\rho)\zeta' \otimes c_{kl}(u) \cdot \zeta \Big), \end{aligned}$$

by definition (2.2.44); here we used the notation $\bar{\imath} = n - i + 1$. Assuming induction, $c_{kl}(u)\zeta = 0$. Finally, by (2.2.18) and (2.2.44), we have that

$$\bar{c}_{ik}(u)\bar{d}_{\bar{\jmath}\bar{l}}(-u-\rho)\zeta' = \bar{d}_{\bar{\jmath}\bar{l}}(-u-\rho)\bar{c}_{ik}(u)\zeta' = 0.$$

Hence $c_{ij}(u) \cdot (\zeta' \otimes \zeta) = 0$, as required. Next, we need to show that $a_{ij}(u) \cdot M^0 \subseteq M^0[u^{-1}]$. The base case is given by Lemma 2.2.11. For $\ell \geq 1$ we have

$$\begin{aligned} a_{ij}(u) \cdot (\zeta' \otimes \zeta) &= \sum_{k,l=1}^{n} \Big(\bar{a}_{ik}(u) \bar{d}_{\bar{\jmath}\bar{l}}(-u-\rho)\zeta' \otimes a_{kl}(u) \cdot \zeta \pm \bar{a}_{ik}(u)\bar{c}_{\bar{\jmath}\bar{l}}(-u-\rho)\zeta' \otimes b_{kl}(u) \cdot \zeta \\ &\qquad + \bar{b}_{ik}(u)\bar{d}_{\bar{\jmath}\bar{l}}(-u-\rho)\zeta' \otimes c_{kl}(u) \cdot \zeta \pm \bar{b}_{ik}(u)\bar{c}_{\bar{\jmath}\bar{l}}(-u-\rho)\zeta' \otimes d_{kl}(u) \cdot \zeta \Big) \\ &= \sum_{k,l=1}^{n} \bar{a}_{ik}(u)\bar{d}_{\bar{\jmath}\bar{l}}(-u-\rho)\zeta' \otimes a_{kl}(u) \cdot \zeta, \end{aligned}$$

by definition (2.2.44) and the result above. Assuming induction, $a_{kl}(u) \cdot \zeta \in M^{(\ell-1)}[u^{-1}]$ and, by Lemma 2.2.12, $\bar{a}_{ik}(u)\bar{d}_{\bar{\jmath}\bar{l}}(-u-\rho)\zeta' \in L^0(\lambda^{(\ell)})_{c_\ell}[u^{-1}]$. Hence $a_{ij}(u) \cdot (\zeta' \otimes \zeta) \in M^0[u^{-1}]$. This proves the claim. □

The last ingredients we will need are the auxiliary spaces $V_{\tilde{a}_i}$ and V_{a_i}. They are vector representations of $\mathfrak{gl}_n$ of weight $\lambda^{(\tilde{a}_i)} = \lambda^{(a_i)} = (1, 0, \ldots, 0)$. Denote by $L^t(\lambda)_c$ the evaluation module of $Y(\mathfrak{gl}_n)$ obtained from the $\mathfrak{gl}_n$-representation $L(\lambda)$ by composing the evaluation map ev_c in (1.2.10) with the algebra automorphism $T(u) \to T^t(-u)$. The spaces $V_{\tilde{a}_i}$ and V_{a_i} can thus be viewed as evaluation modules $L^t(\lambda^{(\tilde{a}_i)})_{-u_i}$ and $L^t(\lambda^{(a_i)})_{u_i}$ of $Y(\mathfrak{gl}_n)$, respectively, with the lowest weights given by

$$\begin{aligned} \lambda_j^{(\tilde a_i)}(u) = \lambda_j^{(a_i)}(u) = 1 \quad \text{for} \quad 1 \le j \le n-1 \quad \text{and} \\ \lambda_n^{(\tilde a_i)}(v) = \frac{v-u_i+1}{v-u_i}, \quad \lambda_n^{(a_i)}(v) = \frac{v+u_i+1}{v+u_i}. \end{aligned} \tag{2.2.47}$$

In particular, the matrix $T_a(v)$ of $Y(\mathfrak{gl}_n)$ acts on the space $L^t(\lambda^{(\tilde a_i)})_{-u_i}$ as $R^t_{a\tilde a_i}(u_i-v)$ and on the space $L^t(\lambda^{(a_i)})_{u_i}$ as $R^t_{aa_i}(-u_i-v)$; here note that $R^t_{ab}(u)=R^t_{ba}(u)$.

We define the *nested vacuum sector* as a tensor product the auxiliary spaces and the vacuum sector M^0:

$$M' := W\otimes M^0, \qquad W = V_{\tilde a_1}\otimes\cdots\otimes V_{\tilde a_m}\otimes V_{a_1}\otimes\cdots\otimes V_{a_m}. \tag{2.2.48}$$

Proposition 2.2.14. *Let $T(v)$ be the generating matrix of $Y(\mathfrak{gl}_n)$. Then the mapping*

$$Y(\mathfrak{gl}_n)\to \mathrm{End}(W)\otimes Y^{\pm}_{\rho}(\mathfrak{gl}_{2n}), \qquad T(v)\mapsto T(v;\boldsymbol{u})$$

equips the space M' with the structure of a lowest weight $Y(\mathfrak{gl}_n)$-module with the lowest weight given by

$$\lambda_i(v;\boldsymbol{u}) = \lambda_i(u)\,\lambda_{2n-i+1}(-u)\,\mu_i(u)\prod_{j=1}^{m}\lambda_i^{(a_j)}(v)\,\lambda_i^{(\tilde a_j)}(v) \tag{2.2.49}$$

for $1\le i\le n$ with $\lambda_i(v)$ defined in (2.1.2), $\mu_i(v)$ in (2.2.5) and $\lambda_i^{(a_j)}(v)$, $\lambda_i^{(\tilde a_j)}(v)$ in (2.2.47).

Proof. It follows from the definition (2.2.42) and Lemma 2.2.13, that the space M' is stable under the action of $T_a(v;\boldsymbol{u})$. Moreover, for any $\zeta\in M'$, we have that

$$R_{ab}(v-w)\,T_a(v;\boldsymbol{u})\,T_b(w;\boldsymbol{u})\cdot\zeta = T_b(w;\boldsymbol{u})\,T_a(v;\boldsymbol{u})\,R_{ab}(v-w)\cdot\zeta.$$

Indeed, we can interleave the matrices on the l.h.s. of the equality above, then use the transposed Yang-Baxter equation to reorder the product of matrices:

$$\begin{aligned} &R_{ab}(v-w)\,T_a(v;\boldsymbol{u})\,T_b(w;\boldsymbol{u}) \\ &\quad = R_{ab}(v-w)\left(\prod_{k=1}^{m} R^t_{\tilde a_k a}(u_k-v)R^t_{\tilde a_k b}(u_k-w)\right)\left(\prod_{l=1}^{m} R^t_{a_l a}(-u_l-v)R^t_{a_l b}(-u_l-w)\right)A_a(v)A_b(w) \\ &\quad = \left(\prod_{k=1}^{m} R^t_{\tilde a_k b}(u_k-w)R^t_{\tilde a_k a}(u_k-v)\right)\left(\prod_{l=1}^{m} R^t_{a_l b}(-u_l-w)R^t_{a_l a}(-u_l-v)\right)R_{ab}(v-w)A_a(v)A_b(w). \end{aligned}$$

From here we use (2.2.22) to obtain the result, plus additional terms. However, $C(u)$ appears as the rightmost operator acting nontrivially on $M^0\subset M'$ in each of these additional terms. Since $C(u)$ annihilates all vectors in M^0, these additional terms vanish. Its lowest vector is

$$\eta := e_{\tilde a_1}\otimes\cdots\otimes e_{\tilde a_m}\otimes e_{a_1}\otimes\cdots\otimes e_{a_m}\otimes\eta_1\otimes\cdots\otimes\eta_\ell\otimes\xi, \tag{2.2.50}$$

where ξ is a lowest vector of $M^0(\mu)$, each η_i is a lowest vector of $L(\lambda^{(i)})^0_{c_i}$ for $1\le i\le \ell$, and each

$e_{\tilde{a}_i}$ (resp. e_{a_i}) is a lowest vector of $V_{\tilde{a}_i}$ (resp. V_{a_i}) for $1 \le i \le m$ (viewed as an evaluation module $L^t(\lambda^{(\tilde{a}_i)})_{-u_i}$ (resp. $L^t(\lambda^{(a_i)})_{-u_i}$)). Finally, acting with $t_{ii}(v;\boldsymbol{u})$ on η for $1 \le i \le n$ and using (2.1.2), (2.2.5) and (2.2.47) yields (2.2.49). □

Remark 2.2.15. *(i)* Recall that $L^0(\lambda^{(i)})_{c_i} \cong L^0(\lambda'^{(i)})_{c_i} \otimes L^0(\lambda''^{(i)})_{c_i}$ with $\overline{A}(u)$ (resp. $\overline{D}^t(u)$) acting non-trivially on the first (resp. second) tensorand only. We may thus rewrite the space M^0 as

$$M^0 \cong L^0(\lambda'^{(1)})_{c_1} \otimes \cdots \otimes L^0(\lambda'^{(\ell)})_{c_\ell} \otimes M^0(\mu) \otimes L^0(\lambda''^{(\ell)})_{c_\ell} \otimes \cdots \otimes L^0(\lambda''^{(1)})_{c_1}.$$

By Proposition 2.2.14, we may view this space as a lowest weight $Y(\mathfrak{gl}_n)$-module. Provided the binary property holds, it is an irreducible $Y(\mathfrak{gl}_n)$-module, see Theorem 6.5.8 in [Mo07]. *(ii)* Enumerate the tensorands of M^0 above by $1, 2, \ldots, 2\ell, 2\ell+1$. Then the matrix $T_a(v;\boldsymbol{u})$ acts on the space $M' = W \otimes M^0$ via the operator

$$\left(\prod_{k=1}^{m} R^t_{a\tilde{a}_k}(u_k - v)\right)\left(\prod_{l=1}^{m} R^t_{aa_l}(-u_l - v - \rho)\right) \\ \times \left(\prod_{i=1}^{\ell} \mathcal{L}^0_{ai}(u - c_i)\right)\mathcal{L}^{\pm 0}_{a,\ell+1}(\mu)\left(\prod_{i=\ell}^{1} \left(\mathcal{L}^0_{a,2\ell-i+1}(-u-\rho-c_i)\right)^t\right),$$

where the Lax operators are those defined in (2.2.45) and (2.2.43).

We end this section with one more technical result which will assist us in finding the explicit expressions of the unwanted terms in Section 2.2.11.

Lemma 2.2.16. *The following identities hold:*

$$\check{R}_{a_i a_{i+1}}(u_i - u_{i+1})\check{R}^{-1}_{\tilde{a}_i\tilde{a}_{i+1}}(u_i - u_{i+1})\, t_{kl}(w;\boldsymbol{u}) = t_{kl}(w;\boldsymbol{u}_{i\leftrightarrow i+1})\check{R}_{a_i a_{i+1}}(u_i - u_{i+1})\check{R}^{-1}_{\tilde{a}_i\tilde{a}_{i+1}}(u_i - u_{i+1}).$$
$$\check{R}_{a_i a_{i+1}}(u_i - u_{i+1})\check{R}^{-1}_{\tilde{a}_i\tilde{a}_{i+1}}(u_i - u_{i+1})\,\eta = \eta.$$

Proof. The first identity is achieved by moving the $\check{R}$-matrices through each matrix in the definition of the nested monodromy matrix. Indeed, the $\check{R}$-matrices each commute with all but a pair of adjacent R-matrices in (2.2.42), for which we use the Yang Baxter equations,

$$\begin{aligned} &\check{R}_{a_i a_{i+1}}(u_i - u_{i+1})R^t_{a_i a}(-u_i - v)R^t_{a_{i+1}a}(-u_{i+1} - v) \\ &\qquad\qquad\qquad = R^t_{a_i a}(-u_{i+1} - v)R^t_{a_{i+1}a}(-u_i - v)\check{R}_{a_i a_{i+1}}(u_i - u_{i+1}), \\ &\check{R}^{-1}_{\tilde{a}_i\tilde{a}_{i+1}}(u_i - u_{i+1})R^t_{\tilde{a}_i a}(u_i - v)R^t_{\tilde{a}_{i+1}a}(u_{i+1} - v) = R^t_{\tilde{a}_i a}(u_{i+1} - v)R^t_{\tilde{a}_{i+1}a}(u_i - v)\check{R}^{-1}_{\tilde{a}_i\tilde{a}_{i+1}}(u_i - u_{i+1}), \end{aligned}$$

and the result follows.

To see why the second identity is true, notice that the lowest weight vector η (2.2.50) is an eigenvector of $P_{a_i a_{i+1}}$, and therefore also of $\check{R}_{a_i a_{i+1}}(u_i - u_{i+1})$. This is true also for $P_{\tilde{a}_i\tilde{a}_{i+1}}$. Thus, acting with both $\check{R}_{a_i a_{i+1}}(u_i - u_{i+1})$ and $\check{R}^{-1}_{\tilde{a}_i\tilde{a}_{i+1}}(u_i - u_{i+1})$, the eigenvalues cancel, which gives the result. □

2.2.8 The twisted Yangian spin chain and transfer matrix

We are now ready to consider the nested algebraic Bethe ansatz for a one-dimensional spin chain with open boundary conditions and having twisted Yangian $Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})$ as its underlying symmetry. The full quantum space is the lowest weight $Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})$-module M defined in (2.2.6):

$$M = L(\lambda^{(1)})_{c_1} \otimes L(\lambda^{(2)})_{c_2} \otimes \cdots \otimes L(\lambda^{(\ell)})_{c_\ell} \otimes M(\mu).$$

The generating matrix $S(u)$ of $Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})$ acts on this space via a product of Lax operators (2.2.7):

$$S_a(v) \cdot M = \left(\prod_{i=1}^{\ell} \mathcal{L}_{ai}(v - c_i) \right) \mathcal{L}^{\pm}_{a,\ell+1}(v) \left(\prod_{i=\ell}^{1} \mathcal{L}^t_{ai}(-v-\rho-c_i) \right) M.$$

Taking the trace of the generating matrix we obtain a double-row transfer matrix

$$\tau(v) := \operatorname{tr} S(v) = \operatorname{tr} A(v) + \operatorname{tr} D(v) = \operatorname{tr} A(v) + \operatorname{tr} D^t(v). \tag{2.2.51}$$

One can show using the usual methods that $[\tau(u), \tau(v)] = 0$; see Section 2 in [ACDFR06], also [Sk88]. We seek an eigenvector of $\Psi \in M$ of $\tau(v)$, which we will refer to as the *Bethe vector*. The problem of finding an eigenvector of the transfer matrix (2.2.51) can be substantially simplified with the help of the symmetry relation (2.2.24) which allows us to write the transfer matrix $\tau(v)$ in a symmetric form

$$\tau(v) = p(v) \operatorname{tr} A(v) + p(-v-\rho) \operatorname{tr} A(-v-\rho) = \{p(v) \operatorname{tr} A(v)\}^v,$$

where $p(v)$ is given by (2.2.27). Here we used the notation introduced in (2.2.38). It will therefore be sufficient to focus on the action of $A(v)$, without needing to consider $D(v)$.

The last ingredient we will need is the *nested transfer matrix*, see (2.2.42):

$$t(v; \boldsymbol{u}) := \operatorname{tr} T(v; \boldsymbol{u}).$$

It will play the role of $\tau(v)$ at the nested level of the ansatz. Since we will only consider the action of $T(v;\boldsymbol{u})$ on a finite-dimensional vector space, we can thus specialize the parameters u_i of m-tuple $\boldsymbol{u}$ to nonzero complex numbers. Hence we will further assume that $\boldsymbol{u} \in \mathbb{C}^m$ is an m-tuple of distinct nonzero complex numbers.

Remark 2.2.17. Open spin chains of this type, with "soliton non-preserving" open boundary conditions, were first considered in [Do00], where a construction of a Hamiltonian with local interactions is also given. To specialise to this case, we first choose the number of sites ℓ to be even, and choose the weights $\lambda^{(2i-1)} = (1, 0, \ldots, 0)$ and $\lambda^{(2i)} = (0, 0, \ldots, -1)$, arriving at the "alternating chain" of fundamental and anti-fundamental sites. We also choose the one-dimensional representation of $\mathfrak{so}_{2n}$ or $\mathfrak{sp}_{2n}$ for the boundary site $M(\mu)$. Finally, we fix $\rho = -n$, and set each of the shifts c_i to be zero. The resulting Hamiltonian contains interactions with a range no further than the four nearest sites.

2.2.9 Bethe vector for a single excitation

To introduce the nesting technique, we start by constructing the Bethe vector with a single excitation, i.e., $m = 1$, as this case allows us to expose the main idea of our approach while keeping the technical difficulties to the minimum; for example, in this case we find the unwanted terms without additional computations. Recall the definition of the vacuum sector M^0 (2.2.46) and the nested vacuum sector M' (2.2.48). For $m = 1$ we have $M' = V_{\tilde{a}_1} \otimes V_{a_1} \otimes M^0$. Let $\Phi \in M'$, which we will refer to as the *nested Bethe vector*. The vector Φ may depend on $u \in \mathbb{C}$, hence we will write $\Phi = \Phi(u)$. Using the creation operators defined in Definition 2.2.3, we write an ansatz for the Bethe vector with a single excitation

$$\Psi(u) := \beta_{\tilde{a}_1 a_1}(u) \cdot \Phi(u) \in M. \tag{2.2.52}$$

We now compute the action of the transfer matrix $\tau(v)$ on this Bethe vector. Using Lemma 2.2.8 we have

$$\begin{aligned}
&\tau(v) \cdot \Psi(u) \\
&\quad = \{p(v)\,\mathrm{tr}\,A(v)\}^v \,\beta_{\tilde{a}_1 a_1}(u) \cdot \Phi(u) \\
&\quad = \beta_{\tilde{a}_1 a_1}(u)\,\mathrm{tr}_a \left(\left\{ p(v)\, R^t_{\tilde{a}_1 a}(u-v)\, R^t_{a_1 a}(-u-v-\rho)\, A_a(v) \right\}^v \right) \cdot \Phi(u) \\
&\qquad + \frac{1}{p(u)} \left\{ \frac{p(v)}{u-v} \beta_{\tilde{a}_1 a_1}(v) \right\}^v \operatorname*{Res}_{w \to u}\, \mathrm{tr}_a \left(\left\{ p(w) R^t_{\tilde{a}_1 a}(u-w)\, R^t_{a_1 a}(-u-w-\rho) A_a(w) \right\}^w \right) \cdot \Phi(u) \\
&\quad = \beta_{\tilde{a}_1 a_1}(u)\, \{p(v)\, t(v;u)\}^v \cdot \Phi(u) \\
&\qquad + \frac{1}{p(u)} \left\{ \frac{p(v)}{u-v} \beta_{\tilde{a}_1 a_1}(v) \right\}^v \operatorname*{Res}_{w \to u}\, \{p(w)\, t(w;u)\}^w \cdot \Phi(u).
\end{aligned} \tag{2.2.53}$$

The first term in the r.h.s. of the equality above is the wanted term, as the parameter carried by $\beta_{\tilde{a}_1 a_1}(u)$ is unchanged. The second term has $\beta_{\tilde{a}_1 a_1}(v)$ and is the unwanted term, which we will require to vanish.

Let us now make the additional requirement, which we will justify later, that vector $\Phi(u)$ is an eigenvector of the nested transfer matrix $t(v;u)$ with an eigenvalue $\Gamma(v;u)$:

$$t(v;u) \cdot \Phi(u) = \Gamma(v;u)\, \Phi(u). \tag{2.2.54}$$

This allows us to rewrite (2.2.53) as

$$\begin{aligned}
\tau(v) \cdot \Psi(u) &= \{p(v)\, \Gamma(v;u)\}^v\, \Psi(u) + \frac{1}{p(u)} \operatorname*{Res}_{w \to u}\, \{p(w)\, \Gamma(w;u)\}^w \left\{ \frac{p(v)}{u-v} \beta_{\tilde{a}_1 a_1}(v) \right\}^v \cdot \Phi(u) \\
&= \Lambda(v;u)\, \Psi(u) + \operatorname*{Res}_{w \to u} \Lambda(w;u)\, \frac{1}{p(u)} \left\{ \frac{p(v)}{u-v} \beta_{\tilde{a}_1 a_1}(v) \right\}^v \cdot \Phi(u),
\end{aligned}$$

where $\Lambda(v;u) := \{p(v)\Gamma(v;u)\}^v$. We thus conclude that $\Phi(u)$ is an eigenvector of $\tau(v)$ with eigenvalue $\Lambda(v;u)$ if

$$\operatorname*{Res}_{w\to u} \Lambda(w;u) = 0.$$

This is the *Bethe equation* for u, solutions of which, by (2.2.52), give a set of possible eigenvectors of $\tau(v)$.

It remains to find a nested Bethe vector $\Phi(u)$ satisfying (2.2.54): we seek an eigenvector $\Phi(u) \in M'$ of $t(v;u)$. By Proposition 2.2.14, the nested monodromy matrix $T_a(v;u)$ and the nested vacuum sector M' form a $Y(\mathfrak{gl}_n)$-system. The spectral problem of this system can be solved by means of the usual nested algebraic Bethe ansatz presented in [KR83], which we have recalled in detail in Appendix 2.1. For example, the ansatz for $\Phi(u)$ has the form

$$\Phi(u) = \Phi(\boldsymbol{u}';u) := B'_{a'_1}(u'_1)\cdots B'_{a'_{m'}}(u'_{m'}) \cdot \Phi'(\boldsymbol{u}';u),$$

where $\boldsymbol{u}' = (u'_1,\ldots,u'_{m'}) \in \mathbb{C}^{m'}$ and $B'_{a'_j}(u'_j)$ are creation operators taken from the $T_a(v;u)$. Continuing this nesting procedure, we obtain an eigenvector $\Phi(u;\boldsymbol{u}')$ with eigenvalue, see (2.1.27),

$$\begin{aligned}\Gamma(v;u) = \lambda_1(v;u)\prod_{i=1}^{m'}\frac{v-u_i^{(1)}+1}{v-u_i^{(1)}} + \lambda_n(v;u)\prod_{i=1}^{m^{(n-1)}}\frac{v-u_i^{(n-1)}-1}{v-u_i^{(n-1)}} \\ +\sum_{k=2}^{n-1}\lambda_k(v;u)\prod_{i=1}^{m^{(k-1)}}\frac{v-u_i^{(k-1)}-1}{v-u_i^{(k-1)}}\prod_{j=1}^{m^{(k)}}\frac{v-u_j^{(k)}+1}{v-u_j^{(k)}},\end{aligned}$$

where $\lambda_k(v;u)$ are given by (2.2.14) and the $u_i^{(k)}$ with $1 \le k \le n-1$ are parameters introduced at level k of nesting when diagonalizing the $\mathfrak{gl}_n$-symmetric periodic spin chain. These parameters are fixed to be solutions of their respective Bethe equations, given in (2.1.28).

The boundary eigenvalue $\Lambda(v;u)$ and Bethe equation for u can then be found by substituting the values for $\lambda_k(v;u)$ from (2.2.49) into the above expression. These are given explicitly for multiple excitations by Theorem 2.2.18 in Section 2.2.12.

2.2.10 Bethe vector for multiple excitations

For multiple excitations the argument proceeds similarly to the previous section. Recall that $m \in \mathbb{N}$ is the excitation number and $\boldsymbol{u} \in \mathbb{C}^m$ is an m-tuple of distinct nonzero complex parameters. Let Φ, the *nested Bethe vector*, be a vector from the lowest weight $Y(\mathfrak{gl}_n)$-module M' defined in (2.2.48):

$$\Phi \in M' = V_{\tilde{a}_1} \otimes \cdots \otimes V_{\tilde{a}_m} \otimes V_{a_1} \otimes \cdots \otimes V_{a_m} \otimes M^0.$$

The vector Φ may also depend on the parameters $\boldsymbol{u}$, and we will write $\Phi = \Phi(\boldsymbol{u})$. From the nested Bethe vector, we make the following ansatz for the full Bethe vector:

$$\Psi(\boldsymbol{u}) := \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(\boldsymbol{u}) \cdot \Phi(\boldsymbol{u}) \in M. \tag{2.2.55}$$

We now act with the transfer matrix $\tau(v)$ on this Bethe vector. Using Corollary 2.2.10 we find

$$\tau(v) \cdot \Psi(\boldsymbol{u}) = \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(\boldsymbol{u}) \, \{p(v) \, t(v; \boldsymbol{u})\}^v \cdot \Phi(\boldsymbol{u}) + UWT.$$

The unwanted terms UWT are less simple than in the $m = 1$ case, and will be discussed in detail in the section below. With the expectation that the $\boldsymbol{u}$ may be chosen such that the unwanted terms vanish, the Bethe vector $\Phi(\boldsymbol{u})$ will be an eigenvector of $\tau(v)$ if we take the additional requirement, as for $m = 1$, that

$$t(v; \boldsymbol{u}) \cdot \Phi(\boldsymbol{u}) = \Gamma(v; \boldsymbol{u}) \, \Phi(\boldsymbol{u}). \tag{2.2.56}$$

We therefore seek an eigenvector $\Phi(\boldsymbol{u}) \in M'$ of $t(v; \boldsymbol{u})$. This is found again by the algebraic Bethe ansatz for $Y(\mathfrak{gl}_n)$ with the full quantum space M' and monodromy matrix $T(v; \boldsymbol{u})$, precisely as given in Section 2.1.

From here, proceeding as we did in the $m = 1$ case, we have that

$$\tau(v) \cdot \Psi(\boldsymbol{u}) = \Lambda(v; \boldsymbol{u}) \, \Psi(\boldsymbol{u}) + UWT, \quad \text{where} \quad \Lambda(v; \boldsymbol{u}) = \{p(v) \Gamma(v; \boldsymbol{u})\}^v. \tag{2.2.57}$$

2.2.11 Dealing with unwanted terms

In this section, we will find an exact expression for the unwanted terms from the action of $\tau(v)$ on the Bethe vector and, by setting these terms to zero, we will obtain the Bethe equations.

We begin by introducing some notation for the unwanted terms. Let $U(v; \boldsymbol{u})$ denote the terms initially excluded from the expression in

$$\tau(v) \, \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(\boldsymbol{u}) = \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(\boldsymbol{u}) \, \{p(v) \, t(v; \boldsymbol{u})\}^v + U(v; \boldsymbol{u}).$$

To find an expression for $U(v; \boldsymbol{u})$, begin by acting on $\beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(\boldsymbol{u})$. By repeated applications of (2.2.39), as in Lemma 2.2.9, we may move $A_a(\cdot)$ through each of the remaining creation operators in $\beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(\boldsymbol{u})$, generating a sum of terms in which the rightmost operator is a matrix element of $A_a(u)$ for $u \in \{v, u_1, \ldots, u_m, -v - \rho, -u_1 - \rho, \ldots, -u_m - \rho\}$. To find a more concise expression for $U(v; \boldsymbol{u})$, it will be useful to partition the terms by the parameter appearing in $A_a(\cdot)$. Let $\mathcal{B}$ denote the subalgebra of $Y_\rho^\pm(\mathfrak{gl}_{2n})$ generated by coefficients of $b_{ij}(u)$ for $1 \leq i, j \leq n$, the closure of which is guaranteed by (2.2.21). Then

$$U(v; \boldsymbol{u}) = \sum_{j=1}^{m} \Big(U_{+,j}(v; \boldsymbol{u}) + U_{-,j}(v; \boldsymbol{u}) \Big),$$

where

$$U_{+,j}(v; \boldsymbol{u}) = \sum_{k,l=1}^{n} B_{+,j,kl} \, a_{kl}(u_j), \qquad U_{-,j}(v; \boldsymbol{u}) = \sum_{k,l=1}^{n} B_{-,j,kl} \, a_{kl}(-u_j - \rho)$$

for some $B_{\pm,j,kl} \in \mathcal{B} \otimes (\mathbb{C}^n)^{\otimes 2m}$. Additionally, let us define $U_j(v; \boldsymbol{u}) := U_{+,j}(v; \boldsymbol{u}) + U_{-,j}(v; \boldsymbol{u})$. We will now proceed to find $U_1(v; \boldsymbol{u})$ using the standard techniques. Indeed, consider moving $\tau(v)$

through only the first creation operator. From (2.2.39),

$$\begin{aligned}
&\tau(v)\,\beta_{\tilde{a}_1 a_1\ldots\tilde{a}_m a_m}(\boldsymbol{u})\\
&\quad=\Bigg(\beta_{\tilde{a}_1 a_1}(u_1)\,\mathrm{tr}_a\left\{p(v)R^t_{\tilde{a}_1 a}(u_1-v)R^t_{a_1 a}(-u_1-v-\rho)A_a(v)\right\}^v\\
&\qquad\quad+\frac{1}{p(u_1)}\left\{\frac{p(v)}{u_1-v}\beta_{\tilde{a}_1 a_1}(v)\right\}^v\underset{w\to u_1}{\mathrm{Res}}\ \mathrm{tr}_a\left\{p(w)R^t_{\tilde{a}_1 a}(u_1-w)R^t_{a_1 a}(-u_1-w-\rho)A_a(w)\right\}^w\Bigg)\\
&\qquad\times\prod_{i=2}^{m}\Bigg(\beta_{\tilde{a}_i a_i}(u_i)\prod_{j=i-1}^{1}R_{a_j\tilde{a}_i}(-u_j-u_i-\rho)\Bigg).
\end{aligned}$$

We focus on the second term here, which, upon taking the residue, contains $A_a(u_1)$ and $A_a(-u_1-\rho)$. As all the entries of the m-tuple $\boldsymbol{u}$ are distinct, all contributions to $U_1(v;\boldsymbol{u})$ must originate from moving $A_a(u_1)$ and $A_a(-u_1-\rho)$ through the remaining creation operators without any further parameter exchanges. Therefore, by repeated applications of Lemma 2.2.9,

$$U_1(v;\boldsymbol{u})=\frac{1}{p(u_1)}\left\{\frac{p(v)}{u_1-v}\beta_{\tilde{a}_1 a_1}(v)\right\}^v\prod_{i=2}^{m}\Bigg(\beta_{\tilde{a}_i a_i}(u_i)\prod_{j=i-1}^{1}R_{a_j\tilde{a}_i}(-u_j-u_i-\rho)\Bigg)\underset{w\to u_1}{\mathrm{Res}}\ \{p(w)t(w;\boldsymbol{u})\}^w\,.$$

It now remains to find similar expressions for $U_j(v;\boldsymbol{u})$ for $2\le j\le m$. Recall Lemma 2.2.6. By repeatedly applying such transpositions, we may apply an arbitrary permutation to the parameters $\boldsymbol{u}$ in the m-excitation creation operator. For $\sigma\in\mathfrak{S}_m$, let $\boldsymbol{u}_\sigma$ denote the ordered set $(u_{\sigma(1)},u_{\sigma(2)},\ldots,u_{\sigma(m)})$. Additionally, let σ_j denote the cyclic permutation $\sigma_j:(1,2,\ldots,m)\mapsto(j,j+1,\ldots,j-1)$. We have

$$\beta_{\tilde{a}_1 a_1\ldots\tilde{a}_m a_m}(\boldsymbol{u})=\beta_{\tilde{a}_1 a_1\ldots\tilde{a}_m a_m}(\boldsymbol{u}_{\sigma_j})\,\check{R}_{a_1\ldots a_m}[\sigma_j](\boldsymbol{u})\,\check{R}^{-1}_{\tilde{a}_1\ldots\tilde{a}_m}[\sigma_j](\boldsymbol{u})$$

where $\check{R}[\sigma_j](\boldsymbol{u})$ is the product of $\check{R}$-matrices that generates this permutation. Using this expression for $\beta_{\tilde{a}_1 a_1\ldots\tilde{a}_m a_m}(\boldsymbol{u})$ and following the argument above, we obtain an expression for $U_k(v;\boldsymbol{u})$:

$$\begin{aligned}
U_k(v;\boldsymbol{u})=\frac{1}{p(u_k)}\left\{\frac{p(v)}{u_k-v}\beta_{\tilde{a}_1 a_1}(v)\right\}^v\prod_{i=2}^{m}\Bigg(\beta_{\tilde{a}_i a_i}(u_{\sigma_k(i)})\prod_{j=i-1}^{1}R_{a_j\tilde{a}_i}(-u_{\sigma_k(j)}-u_{\sigma_k(i)}-\rho)\Bigg)&\\
\times\underset{w\to u_k}{\mathrm{Res}}\ \{p(w)\,t(w;\boldsymbol{u}_{\sigma_k})\}^w\,\check{R}_{a_1\ldots a_m}[\sigma_k](\boldsymbol{u})\,\check{R}^{-1}_{\tilde{a}_1\ldots\tilde{a}_m}[\sigma_k](\boldsymbol{u}).&
\end{aligned}$$

By applying this to the nested Bethe vector, we will obtain an expression for all the unwanted terms from the action of $\tau(v)$ on $\Psi(\boldsymbol{u})$. Moving the $\check{R}$ matrices back through the nested transfer matrix by (2.2.16) allows us to use the fact that the nested Bethe vector is assumed to be an eigenvector of the nested transfer matrix. Therefore, summing all these unwanted terms gives

$$
\begin{aligned}
&U(v;\boldsymbol{u})\cdot\Phi(\boldsymbol{u})\\
&\quad=\sum_{k=1}^{m}\frac{1}{p(u_k)}\left\{\frac{p(v)}{u_k-v}\beta_{\tilde{a}_1a_1}(v)\right\}^{v}\prod_{i=2}^{m}\left(\beta_{\tilde{a}_ia_i}(u_{\sigma_k(i)})\prod_{j=i-1}^{1}R_{a_j\tilde{a}_i}(-u_{\sigma_k(j)}-u_{\sigma_k(i)}-\rho)\right)\\
&\qquad\times\check{R}_{a_1\dots a_m}[\sigma_k](\boldsymbol{u})\,\check{R}^{-1}_{\tilde{a}_1\dots\tilde{a}_m}[\sigma_k](\boldsymbol{u})\,\operatorname*{Res}_{w\to u_k}\,\{p(w)t(w;\boldsymbol{u})\}^{w}\cdot\Phi(\boldsymbol{u})\\
&\quad=\sum_{k=1}^{m}\frac{1}{p(u_k)}\operatorname*{Res}_{w\to u_k}\,\{p(w)\Gamma(w;\boldsymbol{u})\}^{w}\left\{\frac{p(v)}{u_k-v}\beta_{\tilde{a}_1a_1}(v)\right\}^{v}\\
&\qquad\times\prod_{i=2}^{m}\left(\beta_{\tilde{a}_ia_i}(u_{\sigma_k(i)})\prod_{j=i-1}^{1}R_{a_j\tilde{a}_i}(-u_{\sigma_k(j)}-u_{\sigma_k(i)}-\rho)\right)\check{R}_{a_1\dots a_m}[\sigma_k](\boldsymbol{u})\,\check{R}^{-1}_{\tilde{a}_1\dots\tilde{a}_m}[\sigma_k](\boldsymbol{u})\cdot\Phi(\boldsymbol{u}).
\end{aligned}
$$

The Bethe equations are then extracted by demanding $U(v;\boldsymbol{u})\cdot\Phi(\boldsymbol{u})=0$. Since each summand is independent, we obtain

$$\operatorname*{Res}_{w\to u_k}\,\{p(w)\Gamma(w;\boldsymbol{u})\}^{w}=0\quad\text{for}\quad 1\le k\le m$$

or, more concisely

$$\operatorname*{Res}_{w\to u_k}\Lambda(w;\boldsymbol{u})=0\quad\text{for}\quad 1\le k\le m. \tag{2.2.58}$$

2.2.12 Boundary eigenvalues and Bethe equations

From the nested algebraic Bethe ansatz for a $Y(\mathfrak{gl}_n)$-system, we have explicit values for the eigenvalues of the nested system, see (2.1.27),

$$
\begin{aligned}
\Gamma(v;\boldsymbol{u})=\lambda_1(v;\boldsymbol{u})\prod_{i=1}^{m^{(1)}}\frac{v-u_i^{(1)}+1}{v-u_i^{(1)}}+\lambda_n(v;\boldsymbol{u})\prod_{i=1}^{m^{(n-1)}}\frac{v-u_i^{(n-1)}-1}{v-u_i^{(n-1)}}\\
+\sum_{k=2}^{n-1}\lambda_k(v;\boldsymbol{u})\prod_{i=1}^{m^{(k-1)}}\frac{v-u_i^{(k-1)}-1}{v-u_i^{(k-1)}}\prod_{i=1}^{m^{(k)}}\frac{v-u_i^{(k)}+1}{v-u_i^{(k)}},
\end{aligned}
$$

where $\lambda_k(v;\boldsymbol{u})$ are given by Proposition 2.2.14. Note that the $(i+1)$-th level of nesting for $Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})$ corresponds to i-th level for $Y(\mathfrak{gl}_n)$. The parameters $u_i^{(k)}$ satisfy the appropriate Bethe equations of $Y(\mathfrak{gl}_n)$ given in (2.1.28). The full eigenvalues $\Lambda(v;\boldsymbol{u})=\{p(v)\Gamma(v;\boldsymbol{u})\}^{v}$ of the Bethe vectors, cf., (2.2.57), are then obtained by substituting our values for $\lambda_k(v;\boldsymbol{u})$ from (2.2.49). This leads to the following statement.

Theorem 2.2.18. *The eigenvalues of the Bethe vectors for a $Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})$-system are given by*

$$\Lambda(v;\boldsymbol{u})=\left(1\pm\frac{1}{2v+\rho}\right)\Gamma(v;\boldsymbol{u})+\left(1\mp\frac{1}{2v+\rho}\right)\Gamma(-v-\rho;\boldsymbol{u}), \tag{2.2.59}$$

where

$$\begin{aligned}\Gamma(v;\boldsymbol{u}) = &\left(\prod_{j=1}^{\ell}\frac{v-c_j-\lambda_1^{(j)}}{v-c_j}\cdot\frac{v+\rho+c_j+\lambda_{2n}^{(j)}}{v+\rho+c_j}\right)\left(\prod_{i=1}^{m^{(1)}}\frac{v-u_i^{(1)}+1}{v-u_i^{(1)}}\right)\left(\frac{v+(\rho\pm 1)/2-\mu_1}{v+(\rho\pm 1)/2}\right)\\
&+\left(\prod_{j=1}^{\ell}\frac{v-c_j-\lambda_n^{(j)}}{v-c_j}\cdot\frac{v+\rho+c_j+\lambda_{n+1}^{(j)}}{v+\rho+c_j}\right)\left(\prod_{i=1}^{m}\frac{v-u_i+1}{v-u_i}\cdot\frac{v+\rho+u_i+1}{v+\rho+u_i}\right)\\
&\times\left(\prod_{i=1}^{m^{(n-1)}}\frac{v-u_i^{(n-1)}-1}{v-u_i^{(n-1)}}\right)\left(\frac{v+(\rho\pm 1)/2-\mu_n}{v+(\rho\pm 1)/2}\right)\\
&+\sum_{k=2}^{n-1}\left(\prod_{j=1}^{\ell}\frac{v-c_j-\lambda_k^{(j)}}{v-c_j}\cdot\frac{v+\rho+c_j+\lambda_{2n-k+1}^{(j)}}{v+\rho+c_j}\right)\\
&\times\left(\prod_{i=1}^{m^{(k-1)}}\frac{v-u_i^{(k-1)}-1}{v-u_i^{(k-1)}}\right)\left(\prod_{i=1}^{m^{(k)}}\frac{v-u_i^{(k)}+1}{v-u_i^{(k)}}\right)\left(\frac{v+(\rho\pm 1)/2-\mu_k}{v+(\rho\pm 1)/2}\right).\end{aligned}\tag{2.2.60}$$

By (2.2.58), the Bethe equations for $\boldsymbol{u}$ are found by demanding that the residue of the eigenvalue (2.2.59) vanishes at each of the u_k with $1\le k\le m$. Evaluating this residue and using the fact that the $Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})$-system reduces to a $Y(\mathfrak{gl}_n)$-system we obtain the following.

Theorem 2.2.19. *The Bethe equations for a $Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})$-system are given by* (2.1.28) *and*

$$\begin{aligned}&\frac{u_k+(\rho-1)/2}{u_k+(\rho+1)/2}\cdot\frac{u_k+(\rho\mp 1)/2+\mu_n}{u_k+(\rho\pm 1)/2-\mu_n}\left(\prod_{i\neq k}\frac{u_k-u_i-1}{u_k-u_i+1}\cdot\frac{u_k+u_i+\rho-1}{u_k+u_i+\rho+1}\right)\\
&=\left(\prod_{j=1}^{\ell}\frac{u_k-c_j-\lambda_n^{(j)}}{u_k-c_j-\lambda_{n+1}^{(j)}}\cdot\frac{u_k+\rho+c_j+\lambda_{n+1}^{(j)}}{u_k+\rho+c_j+\lambda_n^{(j)}}\right)\left(\prod_{i=1}^{m^{(n-1)}}\frac{u_k+\rho+u_i^{(n-1)}}{u_k+\rho+u_i^{(n-1)}+1}\cdot\frac{u_k-u_i^{(n-1)}-1}{u_k-u_i^{(n-1)}}\right)\end{aligned}\tag{2.2.61}$$

for $1\le k\le m$.

Remark 2.2.20. The condition (2.1.26) is equivalent to the vanishing of the residue of $\Lambda(v;\boldsymbol{u})$ in (2.2.59) at each of the $u_i^{(k)}$, which is the expected Bethe equation for a system of equations.

Remark 2.2.21. The eigenvalue $\Lambda(v;\boldsymbol{u})$ for a $Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})$-system, when the evaluation module $M(\mu)$ of $Y^{\pm}_{\rho}(\mathfrak{gl}_{2n})$ in (2.2.6) is a one-dimensional, was calculated in [ACDFR06] by means of the analytical Bethe ansatz. By shifting the roots of the equations and including the assumption that the roots come in pairs, one can recover the eigenvalue found in [ACDFR06] from (2.2.60) and (2.2.59).

2.2.13 A trace formula for the Bethe vectors

Recall the trace formula for the Bethe vectors for the closed $\mathfrak{gl}_n$-symmetric spin chain (2.1.29). This formula allows us to write an expression of the nested Bethe vector of the residual $Y(\mathfrak{gl}_n)$-system in terms of a trace of elements of the nested monodromy matrix as follows:

$$\begin{aligned}
&\Phi_{a_1,\dots,a_m,\tilde a_1,\dots,\tilde a_m}(\boldsymbol{u},\boldsymbol{u}^{(1)},\dots,\boldsymbol{u}^{(n-1)}) \\
&\quad = \mathrm{tr}_{\overline{V}}\left[\left(\prod_{k=1}^{n-1}\prod_{i=1}^{m^{(k)}} T_{a_i^k}(u_i^{(k)},\boldsymbol{u})\right)\left(\prod_{k=2}^{n-1}\prod_{l=1}^{k-1}\prod_{i=1}^{m^{(k)}}\prod_{j=m^{(l)}}^{1} R_{a_i^k a_j^l}(u_i^{(k)}-u_j^{(l)})\right)\right. \\
&\qquad\qquad \left.\times (e_{21})^{\otimes m^{(1)}}\otimes\cdots\otimes(e_{n,n-1})^{\otimes m^{(n-1)}}\right]\cdot\eta, \qquad (2.2.62)
\end{aligned}$$

where the trace is taken over the space $\overline{V} := V_{a_1^1}\otimes\cdots\otimes V_{a^{n-1}_{m^{(n-1)}}} \cong (\mathbb{C}^n)^{\otimes\overline{m}}$ with $\overline{m} = \sum_{i=1}^{n-1} m^{(i)}$ and η is the lowest vector (2.2.50) of the nested vacuum sector M', c.f. (2.2.48). Our goal is to extended this formula for Bethe vectors (2.2.55) of the $Y_\rho^\pm(\mathfrak{gl}_{2n})$-system.

Recall the notation from Section 2.2.2. The R-matrix acting on $\mathbb{C}^{2n}\otimes\mathbb{C}^{2n}$ is denoted by $\mathbb{R}(u)$ and the matrix units of $\mathrm{End}(\mathbb{C}^{2n})$ (resp. $\mathrm{End}_U(\mathbb{C}^2)$) by e_{ij} for $1\le i,j\le 2n$ (resp. x_{ij} for $1\le i,j\le 2$). We will use symbols W_a (W_{a_i}, $W_{\tilde a_i}$, $W_{a_i^k}$, ...) to denote copies of $\mathbb{C}^{2n}$; symbols V_a (V_{a_i}, $V_{\tilde a_i}$, $V_{a_i^k}$, ...) will be reserved for copies of $\mathbb{C}^n$. When necessary, we will write $(\mathsf{e}_{ij})_a$ to indicate that $\mathsf{e}_{ij}\in\mathrm{End}(W_a)$, and similarly for $(x_{ij})_a$ and $(e_{ij})_a$; here recall (2.2.10).

Proposition 2.2.22. *The Bethe vector for the $Y_\rho^\pm(\mathfrak{gl}_{2n})$-system can be written as*

$$\begin{aligned}
&\Psi(\boldsymbol{u},\boldsymbol{u}^{(1)},\dots,\boldsymbol{u}^{(n-1)}) \\
&\quad = \mathrm{tr}_{\overline{W}}\left[\prod_{l=1}^{m}\left(\left(\prod_{j=1}^{l-1}\mathbb{R}^t_{a_j a_l}(-u_j-u_l-\rho)\right)\hat S_{a_l}(u_l;\boldsymbol{u}^{(1)},\dots,\boldsymbol{u}^{(n-1)})\right)\left(\prod_{k=1}^{n-1}\prod_{i=1}^{m^{(k)}} S_{a_i^k}(u_i^{(k)})\right)\right. \\
&\qquad \left.\times\left(\prod_{k=2}^{n-1}\prod_{l=1}^{k-1}\prod_{i=1}^{m^{(k)}}\prod_{j=m^{(l)}}^{1}\mathbb{R}_{a_i^k a_j^l}(u_i^{(k)}-u_j^{(l)})\right)(\mathsf{e}_{n+1,n})^{\otimes m}\otimes(\mathsf{e}_{21})^{\otimes m^{(1)}}\otimes\cdots\otimes(\mathsf{e}_{n,n-1})^{\otimes m^{(n-1)}}\right]\cdot\xi, \\
&\hfill (2.2.63)
\end{aligned}$$

where the trace is taken over the space $\overline{W} := W_{a_1}\otimes\cdots\otimes W_{a_m}\otimes W_{a_1^1}\otimes\cdots\otimes W_{a^{n-1}_{m^{(n-1)}}} \cong (\mathbb{C}^{2n})^{\otimes(m+\overline{m})}$ with $\overline{m} = \sum_{i=1}^{n-1} m^{(i)}$, and

$$\hat S_a(u;\boldsymbol{u}^{(1)},\dots,\boldsymbol{u}^{(n-1)}) = \left(\prod_{k=n-1}^{1}\prod_{i=m^{(k)}}^{1}\mathbb{R}_{aa_i^k}(-u-u_i^{(k)}-\rho)\right)S_a(u)\left(\prod_{k=1}^{n-1}\prod_{i=1}^{m^{(k)}}\mathbb{R}^t_{aa_i^k}(u-u_i^{(k)})\right), \qquad (2.2.64)$$

and ξ is the lowest vector of the $Y_\rho^\pm(\mathfrak{gl}_{2n})$-module M defined in (2.2.6).

Proof. We start from (2.2.55), with Φ replaced by (2.2.62),

$$
\begin{aligned}
&\Psi(\boldsymbol{u}, \boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(n-1)}) \\
&\quad = \mathrm{tr}_{\overline{V}} \Bigg[\prod_{l=1}^{m} \Bigg(\beta_{\tilde{a}_l a_l}(u_l) \prod_{j=l-1}^{1} R_{a_j \tilde{a}_l}(-u_j - u_l - \rho) \Bigg) \Bigg(\prod_{k=1}^{n-1} \prod_{i=1}^{m^{(k)}} T_{a_i^k}(u_i^{(k)}; \boldsymbol{u}) \Bigg) \\
&\qquad \times \Bigg(\prod_{k=2}^{n-1} \prod_{l=1}^{k-1} \prod_{i=1}^{m^{(k)}} \prod_{j=m^{(l)}}^{1} R_{a_i^k a_j^l}(u_i^{(k)} - u_j^{(l)}) \Bigg) (e_{21})^{\otimes m^{(1)}} \otimes \cdots \otimes (e_{n,n-1})^{\otimes m^{(n-1)}} \Bigg] \cdot \eta.
\end{aligned}
\tag{2.2.65}
$$

The proof shall proceed in two steps. First, we shall rewrite the above formula in terms of the B-block operator, c.f. (2.2.9), rather than creation operators β, which will allow us to introduce a trace over the corresponding auxiliary spaces. Then, from this formula, we will argue that the $n \times n$ matrix operators B, T and R may be replaced by their $2n \times 2n$ counterparts to complete the proof.

Note that, by commuting matrices which act on different spaces, the creation operator and the product of nested monodromy matrices may be reordered as follows:

$$
\prod_{l=1}^{m} \Bigg(\beta_{\tilde{a}_l a_l}(u_l) \prod_{j=l-1}^{1} R_{a_j \tilde{a}_l}(-u_j - u_l - \rho) \Bigg) = \Bigg(\prod_{l=1}^{m} \beta_{\tilde{a}_l a_l}(u_l) \Bigg) \Bigg(\prod_{l=1}^{m} \prod_{j=l-1}^{1} R_{a_j \tilde{a}_l}(-u_j - u_l - \rho) \Bigg)
$$

and

$$
\begin{aligned}
\prod_{k=1}^{n-1} \prod_{i=1}^{m^{(k)}} T_{a_i^k}(u_i^{(k)}; \boldsymbol{u}) &= \prod_{k=1}^{n-1} \prod_{i=1}^{m^{(k)}} \Bigg[\Bigg(\prod_{l=1}^{m} R^t_{\tilde{a}_l a_i^k}(u_l - u_i^{(k)}) \Bigg) \Bigg(\prod_{l=1}^{m} R^t_{a_l a_i^k}(-u_l - u_i^{(k)} - \rho) \Bigg) A_{a_i^k}(u_i^{(k)}) \Bigg] \\
&= \Bigg(\prod_{l=1}^{m} (\boldsymbol{R}_{\tilde{a}_l}(u_l))^{t_{\tilde{a}_l}} \Bigg) \Bigg(\prod_{l=1}^{m} (\boldsymbol{R}_{a_l}(-u_l - \rho))^{t_{a_l}} \Bigg) \Bigg(\prod_{k=1}^{n-1} \prod_{i=1}^{m^{(k)}} A_{a_i^k}(u_i^{(k)}) \Bigg),
\end{aligned}
$$

where we have introduced

$$
\boldsymbol{R}_a(u) = \prod_{k=n-1}^{1} \prod_{i=m^{(k)}}^{1} R_{a a_i^k}(u - u_i^{(k)}). \tag{2.2.66}
$$

Dependence on $\boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(n-1)}$ has been omitted for clarity. Note also that, as a product of R-matrices, $\boldsymbol{R}_a(u)$ satisfies the RTT relation

$$
R_{ab}(u - v)\, \boldsymbol{R}_a(u)\, \boldsymbol{R}_b(v) = \boldsymbol{R}_b(v)\, \boldsymbol{R}_a(u)\, R_{ab}(u - v).
$$

Including these new expressions in (2.2.65), we make the following reordering,

$$\left(\prod_{l=1}^{m}\prod_{j=l-1}^{1} R_{a_j\tilde{a}_l}(-u_j-u_l-\rho)\right)\left(\prod_{l=1}^{m}\boldsymbol{R}_{\tilde{a}_l}^{t_{\tilde{a}_l}}(u_l)\right)\left(\prod_{l=1}^{m}\boldsymbol{R}_{a_l}^{t_{a_l}}(-u_l-\rho)\right)$$
$$=\left(\prod_{l=1}^{m}\left(\prod_{j=l-1}^{1} R_{a_j\tilde{a}_l}(-u_j-u_l-\rho)\right)\boldsymbol{R}_{\tilde{a}_l}^{t_{\tilde{a}_l}}(u_l)\right)\left(\prod_{l=1}^{m}\boldsymbol{R}_{a_l}^{t_{a_l}}(-u_l-\rho)\right).$$

We now proceed to make repeated applications of the RTT relation, in a similar manner to the proof of Lemma 2.2.9. For example we have, at the centre of the expression,

$$\left(\prod_{j=m-1}^{1} R_{a_j\tilde{a}_m}(-u_j-u_m-\rho)\right)\boldsymbol{R}_{\tilde{a}_m}^{t_{\tilde{a}_m}}(u_m)\,\boldsymbol{R}_{a_1}^{t_{a_1}}(-u_1-\rho)\cdots\boldsymbol{R}_{a_m}^{t_{a_m}}(-u_m-\rho)$$
$$=\boldsymbol{R}_{a_1}^{t_{a_1}}(-u_1-\rho)\cdots\boldsymbol{R}_{a_{m-1}}^{t_{a_{m-1}}}(-u_{m-1}-\rho)\boldsymbol{R}_{\tilde{a}_m}^{t_{\tilde{a}_m}}(u_m)$$
$$\times\left(\prod_{j=m-1}^{1} R_{a_j\tilde{a}_m}(-u_j-u_m-\rho)\right)\boldsymbol{R}_{a_m}^{t_{a_m}}(-u_m-\rho).$$

Continuing inductively, we obtain the equality

$$\prod_{l=1}^{m}\left(\left(\prod_{j=l-1}^{1} R_{a_j\tilde{a}_l}(-u_j-u_l-\rho)\right)\boldsymbol{R}_{\tilde{a}_l}^{t_{\tilde{a}_l}}(u_l)\right)\left(\prod_{l=1}^{m}\boldsymbol{R}_{a_l}^{t_{a_l}}(-u_l-\rho)\right)$$
$$=\prod_{l=1}^{m}\left(\boldsymbol{R}_{\tilde{a}_l}^{t_{\tilde{a}_l}}(u_l)\left(\prod_{j=l-1}^{1} R_{a_j\tilde{a}_l}(-u_j-u_l-\rho)\right)\boldsymbol{R}_{a_l}^{t_{a_l}}(-u_l-\rho)\right).$$

Therefore, (2.2.65) is equivalent to

$$\Psi(\boldsymbol{u},\boldsymbol{u}^{(1)},\ldots,\boldsymbol{u}^{(n-1)})$$
$$=\mathrm{tr}_{\overline{V}}\left[\prod_{l=1}^{m}\left(\beta_{\tilde{a}_l a_l}(u_l)\,\boldsymbol{R}_{\tilde{a}_l}^{t_{\tilde{a}_l}}(u_l)\left(\prod_{j=l-1}^{1} R_{a_j\tilde{a}_l}(-u_j-u_l-\rho)\right)\boldsymbol{R}_{a_l}^{t_{a_l}}(-u_l-\rho)\right)\left(\prod_{k=1}^{n-1}\prod_{i=1}^{m^{(k)}} A_{a_i^k}(u_i^{(k)})\right)\right.$$
$$\left.\times\left(\prod_{k=2}^{n-1}\prod_{l=1}^{k-1}\prod_{i=1}^{m^{(k)}}\prod_{j=m^{(l)}}^{1} R_{a_i^k a_j^l}(u_i^{(k)}-u_j^{(l)})\right)(e_{21})^{\otimes m^{(1)}}\otimes\cdots\otimes(e_{n,n-1})^{\otimes m^{(n-1)}}\right]\cdot\eta. \qquad (2.2.67)$$

To obtain an expression in terms of the B-block operator (as opposed to the creation operator β), we utilise (2.2.36). Indeed,

$$
\begin{aligned}
&\Psi(\boldsymbol{u},\boldsymbol{u}^{(1)},\dots,\boldsymbol{u}^{(n-1)}) \\
&\quad= \mathrm{tr}_{\overline{V}}\Bigg[\Bigg(\sum_{\substack{r_1,\dots,r_m,\\ s_1,\dots,s_m=1}}^{n}\Bigg(\prod_{l=1}^{m}\Bigg(\Bigg(\prod_{j=1}^{l-1}R^{t}_{a_ja_l}(-u_j-u_l-\rho)\Bigg)\boldsymbol{R}_{a_l}(-u_l-\rho)B_{a_l}(u_l)\boldsymbol{R}^{t_{a_l}}_{a_l}(u_l)\Bigg)\Bigg)_{s_1\bar{r}_1\dots s_m\bar{r}_m} \\
&\qquad\qquad\otimes e^*_{r_1}\otimes\cdots\otimes e^*_{r_m}\otimes e^*_{s_1}\otimes\cdots\otimes e^*_{s_m}\Bigg)\Bigg(\prod_{k=1}^{n-1}\prod_{i=1}^{m^{(k)}}A_{a^k_i}(u^{(k)}_i)\Bigg) \\
&\qquad\times\Bigg(\prod_{k=2}^{n-1}\prod_{l=1}^{k-1}\prod_{i=1}^{m^{(k)}}\prod_{j=m^{(l)}}^{1}R_{a^k_ia^l_j}(u^{(k)}_i-u^{(l)}_j)\Bigg)(e_{21})^{\otimes m^{(1)}}\otimes\cdots\otimes(e_{n,n-1})^{\otimes m^{(n-1)}}\Bigg]\cdot\eta.
\end{aligned}
$$

Recall that $\eta=(e_1)^{\otimes m}\otimes(e_1)^{\otimes m}\otimes\xi$. After contracting the dual vectors with the vector η, the resulting matrix element may then be written in terms of a trace over $\widetilde{V}:=V_{a_1}\otimes\cdots\otimes V_{a_m}\cong(\mathbb{C}^n)^{\otimes m}$, using the identity $(M)_{ji}=\mathrm{tr}(Me_{ij})$. This gives the expression

$$
\begin{aligned}
&\Psi(\boldsymbol{u},\boldsymbol{u}^{(1)},\dots,\boldsymbol{u}^{(n-1)}) \\
&\quad= \mathrm{tr}_{\widetilde{V},\overline{V}}\Bigg[\prod_{l=1}^{m}\Bigg(\Bigg(\prod_{j=1}^{l-1}R^{t}_{a_ja_l}(-u_j-u_l-\rho)\Bigg)\boldsymbol{R}_{a_l}(-u_l-\rho)B_{a_l}(u_l)\,\boldsymbol{R}^{t_{a_l}}_{a_l}(u_l)\Bigg)\Bigg(\prod_{k=1}^{n-1}\prod_{i=1}^{m^{(k)}}A_{a^k_i}(u^{(k)}_i)\Bigg) \\
&\qquad\times\Bigg(\prod_{k=2}^{n-1}\prod_{l=1}^{k-1}\prod_{i=1}^{m^{(k)}}\prod_{j=m^{(l)}}^{1}R_{a^k_ia^l_j}(u^{(k)}_i-u^{(l)}_j)\Bigg)(e_{1n})^{\otimes m}\otimes(e_{21})^{\otimes m^{(1)}}\otimes\cdots\otimes(e_{n,n-1})^{\otimes m^{(n-1)}}\Bigg]\cdot\xi.
\end{aligned}
\tag{2.2.68}
$$

It remains to show that this expression may be rewritten in terms of the original monodromy matrix $S(u)$ and the R-matrix $\mathbb{R}(u)$. We will do this by showing that the expression (2.2.63) reduces to the above expression (2.2.68). We put $\hat{S}_a:=\hat{S}_a(u;\boldsymbol{u}^{(1)},\dots,\boldsymbol{u}^{(n-1)})$ and rewrite the r.h.s. of (2.2.63) as

$$
\begin{aligned}
&\mathrm{tr}_{\overline{W}}\Bigg[\Bigg(\prod_{l=1}^{m}\Bigg(\prod_{j=1}^{l-1}\mathbb{R}^{t}_{a_ja_l}(-u_j-u_l-\rho)\Bigg)\hat{S}_{a_l}\Bigg)(\mathsf{e}_{n+1,n})^{\otimes m} \\
&\qquad\otimes\Big(\hat{\boldsymbol{A}}\hat{\boldsymbol{R}}\Big((\mathsf{e}_{21})^{\otimes m^{(1)}}\otimes\cdots\otimes(\mathsf{e}_{n,n-1})^{\otimes m^{(n-1)}}\Big)\Big)\Bigg]\cdot\xi,
\end{aligned}
\tag{2.2.69}
$$

where operators $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{R}}$ denote the products in the third line of (2.2.63). Recall (2.2.10) and write $(\mathsf{e}_{n+1,n})^{\otimes m}=(x_{21})^{\otimes m}\otimes(e_{1n})^{\otimes m}$ and

$$
(\mathsf{e}_{21})^{\otimes m^{(1)}}\otimes\cdots\otimes(\mathsf{e}_{n,n-1})^{\otimes m^{(n-1)}}=(x_{11})^{\otimes\overline{m}}\otimes(e_{21})^{\otimes m^{(1)}}\otimes\cdots\otimes(e_{n,n-1})^{\otimes m^{(n-1)}}.
$$

From (2.2.12) we see that

$$\mathbb{R}_{a_i^k a_j^l}(u_i^{(k)} - u_j^{(l)})\,(x_{11})_{a_i^k}(x_{11})_{a_j^l} = (x_{11})_{a_i^k}(x_{11})_{a_j^l} R_{a_i^k a_j^l}(u_i^{(k)} - u_j^{(l)})\,.$$

Moreover,

$$S_{a_i^k}(u_i^{(k)})\,(x_{11})_{a_i^k} = (x_{11})_{a_i^k} A_{a_i^k}(u_i^{(k)}) + (x_{21})_{a_i^k} C_{a_i^k}(u_i^{(k)}).$$

Since $C_{a_i^k}(u_i^{(k)})\cdot\xi = 0$, we can neglect the C operator above. Therefore we can replace $\hat{\boldsymbol{A}}\hat{\boldsymbol{R}}\Big((\mathsf{e}_{21})^{\otimes m^{(1)}} \otimes\cdots\otimes(\mathsf{e}_{n,n-1})^{\otimes m^{(n-1)}}\Big)$ in (2.2.69) with $(x_{11})^{\overline{m}}\otimes\Big(\hat{A}\hat{R}\Big((e_{21})^{\otimes m^{(1)}}\otimes\cdots\otimes(e_{n,n-1})^{\otimes m^{(n-1)}}\Big)\Big)$, where operators $\hat{A}$ and $\hat{R}$ denote the operators in the third line of (2.2.68). Now set $\overline{U} := (\mathbb{C}^2)^{\otimes(m+\overline{m})}$ and consider the expression

$$\mathrm{tr}_{\overline{U}}\left[\left(\prod_{l=1}^{m}\left(\prod_{j=1}^{l-1}\mathbb{R}^t_{a_j a_l}(-u_j-u_l-\rho)\right)\hat{S}_{a_l}\right)(x_{21})^{\otimes m}\otimes(x_{11})^{\overline{m}}\right]. \tag{2.2.70}$$

To complete the proof we need to show that the trace above is equivalent to the operators in the second line of (2.2.68). Observe from (2.2.12) that operators $\mathbb{R}$ and $\mathbb{R}^t$ acting on tensor products of x_{11}'s and x_{21}'s preserve their numbers in the tensor product. Hence the trace (2.2.70) is only nonzero when each $\hat{S}_{a_l}$ maps $(x_{21})_{a_l}$ to $(x_{11})_{a_l}$. In particular, using (2.2.12), and the notation (2.2.66), we find that

$$\hat{S}_{a_l}(x_{21})^{\otimes(l)}\otimes(x_{11})^{m-l+\overline{m}} = (x_{21})^{\otimes(l-1)}\otimes(x_{11})^{m-l+1+\overline{m}}\boldsymbol{R}_{a_l}(-u_l-\rho)B_{a_l}(u_l)\boldsymbol{R}^{t_{a_l}}_{a_l}(u_l)+(...),$$

where (...) denotes the terms that do not contribute to the trace. Moreover,

$$\begin{aligned}&\left(\prod_{j=1}^{l-1}\mathbb{R}^t_{a_j a_l}(-u_j-u_l-\rho)\right)(x_{21})^{\otimes(l)}\otimes(x_{11})^{m-l+\overline{m}}\\ &\qquad = (x_{21})^{\otimes(l)}\otimes(x_{11})^{m-l+\overline{m}}\left(\prod_{j=1}^{l-1}R^t_{a_j a_l}(-u_j-u_l-\rho)\right)+(...),\end{aligned}$$

where we have used the same notation as above. Hence the trace (2.2.70) is indeed equivalent to the operators in the second line of (2.2.68). This completes the proof. □

The trace formula (2.2.63) simplifies the process of obtaining the explicit form of the Bethe vectors in terms of the matrix elements of the monodromy matrix $S_a(u)$. We provide here some very basic examples.

Example 2.2.23. Let $n\geq 2$, $m\geq 1$ and $m^{(1)}=\cdots=m^{(n-1)}=0$. Then

$$\Psi(u_1,\ldots,u_m) = \left(\prod_{i=1}^{m}\left(\prod_{j=1}^{i-1}\frac{u_i+u_j+\rho+1}{u_i+u_j+\rho}\right)s_{n,n+1}(u_i)\right)\cdot\xi.$$

Example 2.2.24. Let $n \geq 2$, $m = m^{(i)} = 1$ and $m^{(j)} = 0$ for all $j \neq i$. Then

$$\Psi(u_1, u_1^{(i)}) = \Bigg(s_{n,n+1}(u_1)\, s_{i,i+1}(u_1^{(i)}) \\ + \frac{1}{u_1 - u_1^{(i)}} \left(\frac{u_1 - u_1^{(i)} - 1}{u_1 + u_1^{(i)} + \rho} s_{i,n+1}(u_1) - s_{n,2n-i+1}(u_1) \right) s_{n,i+1}(u_1^{(i)}) \Bigg) \cdot \xi.$$

Example 2.2.25. Let $n \geq 2$, $m = 2$, $m^{(i)} = 1$ and $m^{(j)} = 0$ for all $j \neq i$. Then

$$\begin{aligned} &\Psi(u_1, u_2, u_1^{(i)}) \\ &\quad = \frac{u_1 + u_2 + \rho + 1}{u_1 + u_2 + \rho} \Bigg(s_{n,n+1}(u_1)\, s_{n,n+1}(u_2)\, s_{i,i+1}(u_1^{(i)}) \\ &\quad - \Bigg(\frac{1}{u_2 - u_1^{(i)}} s_{n,n+1}(u_1)\, s_{n,2n-i+1}(u_2) \\ &\qquad - \frac{u_2 - u_1^{(i)} - 1}{u_2 - u_1^{(i)}} \cdot \frac{1}{u_2 + u_1^{(i)} + \rho} \Bigg(s_{n,n+1}(u_1)\, s_{i,n+1}(u_2) + \frac{u_2 + u_1^{(i)} + \rho + 1}{u_1 - u_1^{(i)}} \\ &\qquad\quad \times \left(\frac{u_1 - u_1^{(i)} - 1}{u_1 + u_1^{(i)} + \rho} s_{i,n+1}(u_1) - s_{n,2n-i+1}(u_1) \right) s_{n,n+1}(u_2) \Bigg) s_{n,i+1}(u_1^{(i)}) \Bigg) \Bigg) \cdot \xi. \end{aligned}$$

Remark 2.2.26. Note that in Examples 2.2.24 and 2.2.25 $s_{n,i+1}(u_1^{(i)}) \cdot \xi = 0$ unless $i = n - 1$.

Example: the $\ell = 2$, $Y_\rho^-(\mathfrak{gl}_4)$ chain

In this section we give an analysis of the solutions to the Bethe equations of a simple example system, the $\mathfrak{sp}_4$-symmetric ($n = 2$) chain of length two.

Using the following parameters gives a alternating chain of form studied in [Do00]:

$$\lambda^{(1)} = (1, 0, 0, 0), \qquad c_1 = 0; \qquad \lambda^{(2)} = (0, 0, 0, -1), \qquad c_2 = -\rho; \qquad \mu = (0, 0).$$

Additionally, we must take $\rho = -2$ in order to make use of the Hamiltonian given in the paper.

The $\lambda^{(i)}$ are weights of $\mathfrak{gl}_4$ representations, but the overall $\mathfrak{sp}_4$-symmetry requires us to decompose the representations according to $\mathfrak{sp}_4 \subset \mathfrak{gl}_4$, and we find that both representations above decompose to the $\mathbf{4}$ irrep of $\mathfrak{sp}_4$. As $\mathfrak{sp}_4$-irreps, we can then decompose the full spin chain as follows:

$$\mathbf{4} \otimes \mathbf{4} = \mathbf{10} \oplus \mathbf{5} \oplus \mathbf{1}.$$

Each Bethe vector should correspond to the highest weight vector of one of the above multiplets, with excitation numbers m_i determined by the difference in weights. Specifically, each excitation at level i represents a reduction in the highest weight by root α_i. Then, the multiplicity of the irrep in the decomposition indicates the number of distinct solutions to the Bethe equations with the

$\mathfrak{sp}_4$-irrep	(m_1, m)	$u_1^{(1)}$	$u_2^{(1)}$	u_1	energy
10	$(0,0)$	–	–	–	96
5	$(1,0)$	$1/2$	–	–	-48
		$3/2^*$	–	–	16
1	$(2,1)$	$1/2+i/2$	$1/2-i/2$	$1+i/\sqrt{2}$	-48
		$3/2^*$	$1/4 \mp i/4$	$1.50383 \pm 0.0620246i$	$16 \pm 32i$

Table 2.1: Solutions to the Bethe equations of the $Y_\rho^-(\mathfrak{gl}_4)$ spin chain of length two, with $\rho = -2$. The solutions have been organised according to the corresponding $\mathfrak{sp}_4$ multiplet. Solutions with an asterisk correspond to non-physical states.

given excitation numbers. In practice, the number of solutions of the equations themselves tend to exceed the number predicted by this decomposition, and the desired number is reached only after undergoing a careful selection process.

We present in Table 2.2.13 a list of solutions corresponding to the case $l = 2$. Note that the Bethe vector is invariant under the action of the symmetric group $\mathfrak{S}_{m_i}$ for each level i, so there is a redundancy in the solutions; we have not included the redundant solutions in Table 2.2.13. Further to this, for the top-level there is also a parameter symmetry $u \to -u - \rho$; we have opted to display only the solutions satisfying $\mathrm{Re}(u_1 + \rho/2) > 0$.

We see that, even after accounting for the redundancies, the equations have additional solutions which do not correspond to physical eigenstates of the system. In this case, the parameter $u_1^{(1)}$ satisfies $2u_1^{(1)} + \rho - 1 = 0$, and thus is a zero of the function $p(v)$, which appears in the Bethe ansatz, artificially removing the pole from the eigenvalue.

In this short chain, it is not too difficult to show that the Hamiltonian is given simply by

$$\mathcal{H} = 2\rho(\rho-1)(\rho(\rho \pm 1)P_{12} \pm 2(n+\rho)Q_{12} - \rho) = 24(1 + 2P_{12}).$$

Then, comparing the spectrum of the permutation operator with the above solutions justifies the removal of those solutions with $u_1^{(1)} = 3/2$, as the corresponding energy levels are not present.

In the case of a single excitation, it is a simple calculation to check that the eigenvector obtained by the Bethe ansatz agrees with the results above. Indeed,

$$\begin{aligned}
\Psi(u_1^{(1)}) &= s_{12}(u_1^{(1)}) \cdot \xi \\
&= \left[R_{a1}(u_1^{(1)})R_{a2}^t(-u_1^{(1)} - \rho)R_{a2}(u_1^{(1)})R_{a1}^t(-u_1^{(1)} - \rho)\right]_{12}(e_1 \otimes e_1) \\
&= \left(e_2^*\right)_a R_{a1}(u_1^{(1)})R_{a2}^t(-u_1^{(1)} - \rho)R_{a2}(u_1^{(1)})R_{a1}^t(-u_1^{(1)} - \rho)(e_2 \otimes e_1 \otimes e_1).
\end{aligned}$$

From the last expression above, we see that R_{a1}^t acting on $e_2 \otimes e_1 \otimes e_1$ will act as the identity—the projector Q will only be nonzero if it acts on a tensor product $e_i \otimes e_{2n-i+1}$. Further, since each R-matrix only serves to permute the vectors, we see that R_{a2}^t will also only act as identity. Hence, in this case the Bethe vector is identical to that of a periodic chain of the same length.

Calculating the action of the remaining R-matrices is straightforward, and yields

$$\Psi(u_1^{(1)}) = -\frac{1}{u_1^{(1)}}\left[e_2 \otimes e_1 + \left(1 - \frac{1}{u_1^{(1)}}\right) e_1 \otimes e_2\right].$$

The solution $u_1^{(1)} = 1/2$ gives the antisymmetric state as expected.

Chapter 3

Nested algebraic Bethe ansatz for even orthogonal and symplectic open spin chains

In this chapter we repeat our analysis for an open spin chain constructed from $\mathfrak{so}_{2n}$ or $\mathfrak{sp}_{2n}$ modules, with all possible diagonal boundary conditions. However, as with the previous chapter, we will begin by reviewing a related $\mathfrak{gl}_n$ spin chain, this time the open spin chain with Grassmannian (type AIII) boundary conditions which will appear as the nested system for the orthogonal or symplectic chains. We introduce its underlying reflection algebra $\mathcal{B}^{\mathrm{ex}}_{\rho}(n,p)$ and give the nested algebraic Bethe ansatz for such a chain.

Following this, we introduce the algebra $X_{\rho}(\mathfrak{g}_{2n},\mathfrak{g}^{\theta}_{2n})^{tw}$ and show how it acts on a spin chain constructed from $\mathfrak{so}_{2n}$ or $\mathfrak{sp}_{2n}$ modules. Special consideration must be made when extending these from Lie algebra modules to Yangian modules, as there is no evaluation homomorphism, and we present the fusion procedure for this purpose. We then proceed to give the nested algebraic Bethe ansatz for the system, showing how it reduces to the $\mathfrak{gl}_n$ chain from Section 3.1 . Unlike the twisted Yangian spin chain studied in Chapter 1, care must be taken to separate the orthogonal and symplectic cases.

3.1 Nested algebraic Bethe ansatz for an open $\mathfrak{gl}_n$ spin chain.

3.1.1 The extended reflection algebra $\mathcal{B}^{\mathrm{ex}}_{\rho}(n,p)$

We begin by introducing the extended reflection algebra, following [MR02]. Recall once again the Yangian $Y(\mathfrak{gl}_n)$. In this chapter we will use superscript $\circ$ to distinguish the $Y(\mathfrak{gl}_n)$ generating matrix and weights from those for $X(\mathfrak{g}_{2n})$, so $Y(\mathfrak{gl}_n)$ is generated by matrix $T^{\circ}(u)$ which satisfies the RTT relation with R-matrix $R(u)=1-u^{-1}P$.

In Chapter 1 we introduced the *Ol'shanskii twisted Yangian* $Y^{\pm}(\mathfrak{gl}_{2n})$ as a subalgebra of $Y(\mathfrak{gl}_{2n})$, which led to an open spin chain with $\mathfrak{gl}_{2n}$ bulk symmetry, broken to $\mathfrak{g}_{2n}$ symmetry by the boundary conditions—a type AI(a) or AII symmetric pair. The *(extended) reflection algebra* $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$ corresponds to the type AIII symmetric pair $(\mathfrak{sl}_n, \mathfrak{sl}_p \oplus \mathfrak{sl}_{n-p} \oplus \mathbb{C})$, although in practice it will take the form $(\mathfrak{gl}_n, \mathfrak{gl}_p \oplus \mathfrak{gl}_{n-p})$.

As with the twisted Yangian, the reflection algebra may be defined as a standalone associative algebra generated by a matrix satisfying certain relations, but we will also want to be able to view it as a coideal subalgebra of $Y(\mathfrak{gl}_n)$.

Recall from Table 1.3.2 that symmetric pair AIII is generated by an inner automorphism corresponding to conjugation by an involutory matrix, which we will call G°. The matrix G° takes the form

$$G^\circ = \left(\begin{array}{c|c} I_p & 0 \\ \hline 0 & -I_{n-p} \end{array}\right).$$

We then define the algebra $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$ as follows, including also a 'shift' parameter ρ.

Definition 3.1.1. *The extended reflection algebra $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$ is the unital associative algebra generated by the coefficients of the entries $b^\circ_{ij}(u) = g^\circ_{ij} + \sum_{r\geq 1} b^{\circ(r)}_{ij} u^{-r}$ of the abstract generating matrix $B^\circ(u)$, satisfying the reflection equation*

$$R(u-v)B^\circ(u)R(u+v+\rho)B^\circ(v) = B^\circ(v)R(u+v+\rho)B^\circ(u)R(u-v), \tag{3.1.1}$$

and no other relations.

By the same arguments as in Proposition 2.1 in [MR02], the product $B^\circ(u)\,B^\circ(-u-\rho)$ is a scalar

$$B^\circ(u)\,B^\circ(-u-\rho) = f^\circ(u)\,I, \tag{3.1.2}$$

where $f^\circ(u)$ is an even series with respect to the transformation $u \mapsto -u-\rho$, with coefficients central in $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$. One may then define the *reflection algebra* $\mathcal{B}_\rho(n,p)$ as the quotient of $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$ by the *unitarity relation* $B^\circ(u)\,B^\circ(-u-\rho) = I$. The algebra $\mathcal{B}_\rho(n,p)$ exists as a subalgebra of $Y(\mathfrak{gl}_n)$ generated by

$$T^\circ(u)G^\circ(T^\circ(-u-\rho))^{-1},$$

see [MR02] Theorem 3.1. Then, as a subalgebra of $Y(\mathfrak{gl}_n)$, $\mathcal{B}_\rho(n,p)$ is left-coideal, so

$$\Delta : \mathcal{B}_\rho(n,p) \to Y(\mathfrak{gl}_n) \otimes \mathcal{B}_\rho(n,p).$$

This coideal property is then inherited by the $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$,

$$\Delta(b^\circ_{ij}(u)) = \sum_{k,l=1}^{n} t^\circ_{ik}(u)\,t'^\circ_{lj}(-u-\rho) \otimes b^\circ_{kl}(u).$$

The representation theory of $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$ follows.

Definition 3.1.2. *A representation V of $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$ is called a lowest weight representation if there* *exists a non-zero vector $\eta \in V$ such that $V = \mathcal{B}^{\mathrm{ex}}_{\rho}(n,p)\eta$ and*

$$b^{\circ}_{ji}(u)\eta = 0 \quad \textit{for} \quad 1 \le i < j \le n \quad \textit{and} \quad b^{\circ}_{ii}(u)\eta = \mu^{\circ}_{i}(u)\eta \quad \textit{for} \quad 1 \le i \le n,$$

where $\mu_i(u)$ are formal power series in u^{-1} with constant terms equal to 1 if $i \le n-r$ and -1 if $i > n-r$. The vector η is called the lowest vector of V, and the n-tuple $\mu^{\circ}(u) = (\mu^{\circ}_1(u), \ldots, \mu^{\circ}_n(u))$ is called the lowest weight of V.

We note that any representation V of $\mathcal{B}_{\rho}(n,p)$ may be extended to a representation of $\mathcal{B}^{\mathrm{ex}}_{\rho}(n,p)$ by allowing the series $f^{\circ}(u)$ to act as the identity operator on V.

Recall the action of the inverse matrix $T^{-1}(u)$ on a Yangian $Y(\mathfrak{gl}_n)$ lowest weight module, given by (1.2.17). The following proposition shows that the $\mathcal{B}^{\mathrm{ex}}_{\rho}(n,p)$ modules constructed using lowest weight $Y(\mathfrak{gl}_n)$ modules are themselves lowest weight modules. The proof is given in [GR20a], but is based on that of Proposition 4.10 in [GRW17].

Proposition 3.1.3. *Let η be the lowest vector of a lowest weight $Y(\mathfrak{gl}_n)$-module $L(\lambda(u))$ and let ξ be the lowest vector of a lowest weight $\mathcal{B}^{\mathrm{ex}}_{\rho}(n,p)$-module $V(\mu(u))$. Then $\mathcal{B}^{\mathrm{ex}}_{\rho}(n,p)(\eta \otimes \xi)$ is a lowest weight $\mathcal{B}^{\mathrm{ex}}_{\rho}(n,p)$-module with the lowest vector $\eta \otimes \xi$ and the lowest weight $\gamma^{\circ}(u)$ with components determined by the relations*

$$\widetilde{\gamma}^{\circ}_i(u) = \widetilde{\mu}^{\circ}_i(u)\,\lambda^{\circ}_i(u)\,\lambda'^{\circ}_i(-u-\rho) \quad \textit{for} \quad 1 \le i \le n \tag{3.1.3}$$

with $\widetilde{\mu}^{\circ}_i(u)$ defined by

$$\widetilde{\mu}^{\circ}(u) := (2u + \rho - i + 1)\mu^{\circ}_i(u) + \sum_{j=1}^{i-1} \mu^{\circ}_j(u), \tag{3.1.4}$$

and $\widetilde{\gamma}^{\circ}_i(u)$ defined analogously.

The following proposition rephrases Theorem 3.1 in [BR09] for the algebra $\mathcal{B}^{\mathrm{ex}}_{\rho}(n,p)$, and will be important for the nesting procedure of the Bethe ansatz.

Proposition 3.1.4. *Let $\mathcal{M}$ be a lowest weight $\mathcal{B}^{\mathrm{ex}}_{\rho}(n,p)$-module. For any $1 \le k \le n-1$ define a subspace $\mathcal{M}^{(k)} \subseteq \mathcal{M}$ by*

$$\mathcal{M}^{(k)} := \{v \in \mathcal{M} : b^{\circ}_{ij}(u)v = 0 \textit{ for } i > j \textit{ and } j < k\}.$$

Then operators

$$b^{(k)}_{ij}(u) := b^{\circ}_{ij}\big(u + \tfrac{k-1}{2}\big) + \delta_{ij} \sum_{l=1}^{k-1} \frac{b^{\circ}_{ll}\big(u + \frac{k-1}{2}\big)}{2u + \rho}, \tag{3.1.5}$$

where $k \le i, j \le n$ form a representation of the algebra $\mathcal{B}^{\mathrm{ex}}_{\rho}(n-k+1, r-k+1)$ or $\mathcal{B}^{\mathrm{ex}}_{\rho}(n-k+1, 0)$ in $\mathcal{M}^{(k)}$ for $r > k-1$ or $r \le k-1$, respectively.

Remark 3.1.5. An analogue of Proposition 3.1.4 for the "non-extended" reflection algebra $\mathcal{B}_{\rho}(n,p)$ would require operators $b^{(k)}_{ij}(u)$ in (3.1.5) to be multiplied by a suitable series in u^{-1} with coefficients central in $\mathcal{B}_\rho(n,p)$ to ensure that the corresponding generating matrix $B^{\circ(k)}(u)$ satisfies the unitarity relation in the space $V^{(k)}$.

We now introduce the one-dimensional representations that will function as the boundary of the spin chain. For any $a \in \mathbb{C}$ define a matrix-valued rational function

$$K^\circ(u) = G^\circ - \frac{a}{u + \frac{\rho}{2}} I. \tag{3.1.6}$$

The existence of the free parameter a in this case is a consequence of the fact that the underlying symmetric pair has a nontrivial centre—in this case, we are working with $(\mathfrak{sl}_n, \mathfrak{sl}_r \oplus \mathfrak{sl}_{n-r} \oplus \mathbb{C})$. This K-matrix is a one-parameter solution of the reflection equation (3.1.1) with the R-matrix, and is therefore a representation of $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$. Indeed, more specifically, we have the following Proposition.

Proposition 3.1.6. *(i) Let $r = 0$. The assignment $B^\circ(u) \mapsto I$ yields a one-dimensional representation of $\mathcal{B}^{\mathrm{ex}}_\rho(n,0)$ of weight*

$$\mu^\circ_1(u) = \ldots = \mu^\circ_n(u) = 1. \tag{3.1.7}$$

(ii) Let $1 \le r \le n$. The assignment $B^\circ(u) \mapsto K^\circ(u)$ yields a one-dimensional representation of $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$ of weight $\mu^\circ(u)$ given by

$$\mu^\circ_1(u) = \ldots = \mu^\circ_r(u) = -1 - \frac{a}{u + \frac{\rho}{2}}, \qquad \mu^\circ_{r+1}(u) = \ldots = \mu^\circ_n(u) = 1 - \frac{a}{u + \frac{\rho}{2}}. \tag{3.1.8}$$

We now define the spin chain and the action of $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$ on it. Recall once again the evaluation modules from Section 1.2.1, which in this case will be denoted $L^\circ(\lambda)_c$, where $\lambda = (\lambda_1, \ldots, \lambda_n)$ are the $\mathfrak{gl}_n$ weights of the module, and c is the parameter shift when acted on by $Y(\mathfrak{gl}_n)$ via the evaluation homomorphism. We denote the lowest weight vector of this module by η. Let $V(\mu^\circ)$ denote a $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$ one-dimensional module defined by Proposition 3.1.6.

We will study the spin chain

$$M^\circ = L^\circ \otimes V(\mu^\circ) = L^\circ(\lambda^{(1)})_{c_1} \otimes \cdots \otimes L^\circ(\lambda^{(\ell)})_{c_\ell} \otimes V(\mu^\circ), \tag{3.1.9}$$

which by Proposition 3.1.3 is a lowest weight $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$-module of weight $\gamma^\circ(u)$ with components determined by (recall (3.1.4)),

$$\widetilde{\gamma}^\circ_i(u) = \widetilde{\mu}^\circ_i(u) \prod_{j=1}^{\ell} \lambda^{(j)}_i(u)\, \lambda'^{(j)}_i(u) \quad \text{with} \quad \lambda^{(j)}_i(u) = 1 - \frac{\lambda^{(j)}_i}{u - c_j}. \tag{3.1.10}$$

The action of the generating matrix on the module is given by

$$B^\circ_a(u) \mapsto \mathcal{L}^\circ_{a1}(u - c_1) \cdots \mathcal{L}^\circ_{a\ell}(u - c_\ell) K^\circ_a(u) \big(\mathcal{L}^\circ_{a\ell}(-u - \rho - c_\ell)\big)^{-1} \cdots \big(\mathcal{L}^\circ_{a1}(-u - \rho - c_1)\big)^{-1}. \tag{3.1.11}$$

In what follows we study the spectral problem for a transfer matrix constructed from the above monodromy matrix, acting on the spin chain (3.1.9). Just as in the previous chapter, this will appear as a nested system when studying the even orthogonal and symplectic chains, and so we provide the NABA solution of the $\mathfrak{gl}_n$ system here.

The nesting procedure for this system bears a lot of resemblance to that of the closed $\mathfrak{gl}_n$ spin chain studied in Section 2.1. However, a subtlety is introduced via Proposition 3.1.4—unlike $Y(\mathfrak{gl}_n)$, the $(n-k)\times(n-k)$ submatrix of the generating matrix $B^\circ(u)$ will not itself form a representation of the reduced algebra. Instead, the elements must be shifted in parameter, and diagonal elements of the matrix must be added to obtain a nested system of operators.

The spectral problem for this chain was addressed by Belliard and Ragoucy in [BR09] , thus we will keep this section concise and provide the key steps in the proofs only.

3.1.2 Exchange relations

For any matrix $A=\sum_{i,j=1}^n a_{ij}e_{ij}$ with $e_{ij}\in\mathrm{End}(\mathbb{C}^n)$ and any $1\le k\le n$ define a k-reduced matrix $A^{(k)}=\sum_{i,j=k}^n a_{ij}e^{(k)}_{i-k+1,j-k+1}$ with $e^{(k)}_{ij}\in\mathrm{End}(\mathbb{C}^{n-k+1})$. We use this notation to define k,l-reduced R- and $\check{R}$-matrices acting on the spaces $V_a^{(k)}\cong\mathbb{C}^{n-k+1}$ and $V_b^{(l)}\cong\mathbb{C}^{n-l+1}$ by

$$R^{(k,l)}_{ab}(u):=\frac{u}{u-1}\left(I^{(k,l)}_{ab}-\frac{1}{u}P^{(k,l)}_{ab}\right),\qquad \check{R}^{(k,l)}_{ab}(u):=P^{(k,l)}_{ab}R^{(k,l)}_{ab}(u).$$

Note that $P^{(k,l)}e^{(k)}_i\otimes e^{(l)}_j=0$ if $k<l$ and $i+k-l\le 0$, and $R^{(n,n)}_{ab}(u)$ and $\check{R}^{(n,n)}_{ab}(u)$ are identity operators. We denote the k-reduced generating matrix of $\mathcal{B}^{\rm ex}_\rho(n,p)$ in $\mathrm{End}(V_a^{(k)})$ as $D_a^{(k)}(u)$ and decompose it as

$$D_a^{(k)}(u)=\begin{pmatrix} a^{(k)}(u) & B_a^{(k)}(u)\\ C_a^{(k)}(u) & D_a^{(k+1)}(u)\end{pmatrix}. \tag{3.1.12}$$

We also set

$$\hat{D}_a^{(k)}(u):=D_a^{(k)}\big(u+\tfrac{k-1}{2}\big)+\sum_{i=1}^{k-1}\frac{a^{(i)}\big(u+\frac{k-1}{2}\big)}{2u+\rho}I_a^{(k)}, \tag{3.1.13}$$

$$\hat{a}^{(k)}(u):=a^{(k)}\big(u+\tfrac{k}{2}\big)+\sum_{i=1}^{k-1}\frac{a^{(i)}\big(u+\frac{k}{2}\big)}{2u+1+\rho},\qquad \hat{B}_a^{(k)}(u):=B_a^{(k)}\big(u+\tfrac{k}{2}\big) \tag{3.1.14}$$

leading to the following recursive relations:

$$\big[\hat{D}_a^{(k)}(u)\big]_{ij}=\big[\hat{D}^{(k-1)}\big(u+\tfrac12\big)\big]_{1+i,1+j}+\frac{\delta_{ij}}{2u+\rho}\big[\hat{D}^{(k-1)}\big(u+\tfrac12\big)\big]_{11}, \tag{3.1.15}$$

$$\hat{a}^{(k)}(u)=\big[\hat{D}_a^{(k)}\big(u+\tfrac12\big)\big]_{11}=\big[\hat{D}_a^{(k-1)}(u+1)\big]_{22}+\frac{1}{2u+1+\rho}\hat{a}^{(k-1)}\big(u+\tfrac12\big), \tag{3.1.16}$$

$$\big[\hat{D}_a^{(k)}(u)\big]_{1,1+l}=\big[\hat{B}^{(k)}\big(u-\tfrac12\big)\big]_l \tag{3.1.17}$$

for $1\le i,j\le n-k+1$ and $1\le l\le n-k$. We note that operator $\hat{D}_a^{(k)}(u)$ is a generalisation of Sklyanin's $\widetilde{D}(u)$ operator (see Section 5 in [Sk88]) for arbitrary rank.

Lemma 3.1.7. *Let $\mathcal{M}$ be a lowest weight $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$-module. For any $1 \le k \le n-1$ define a subspace*

$$\mathcal{M}^{(k)} := \{\eta \in \mathcal{M} : b^\circ_{ij}(u)\,\eta = 0 \text{ for } i > j \text{ and } j < k\}. \tag{3.1.18}$$

Let $\equiv$ denote equality of operators in the space $V^{(k)}_a \otimes V^{(k)}_b \otimes \mathcal{M}^{(k)}$. Then

$$\hat{B}^{(k)}_a(v)\,\hat{B}^{(k)}_b(u) \equiv \hat{B}^{(k)}_a(u)\,\hat{B}^{(k)}_b(v)\,\check{R}^{(k+1,k+1)}_{ab}(v-u), \tag{3.1.19}$$

$$\begin{aligned}\hat{a}^{(k)}(v)\,\hat{B}^{(k)}_b(u) \equiv{}& \frac{(v-u+1)(v+u+1+\rho)}{(v-u)(v+u+\rho)}\,\hat{B}^{(k)}_b(u)\,\hat{a}^{(k)}(v)\\ &- \frac{2u+1+\rho}{(v-u)(2u+\rho)}\,\hat{B}^{(k)}_b(v)\,\hat{a}^{(k)}(u) + \frac{1}{v+u+\rho}\hat{B}^{(k)}_b(v)\,\hat{D}^{(k+1)}_b(u),\end{aligned} \tag{3.1.20}$$

$$\begin{aligned}\hat{D}^{(k+1)}_a(v)\,\hat{B}^{(k)}_b(u) \equiv{}& \frac{(v-u-1)(v+u-1+\rho)}{(v-u)(v+u+\rho)}\,\hat{B}^{(k)}_b(u)\,R^{(k+1,k+1)}_{ab}(v+u+\rho)\\ &\qquad\times \hat{D}^{(k+1)}_a(v)\,R^{(k+1,k+1)}_{ab}(v-u)\\ &- \frac{(2v-1+\rho)(2u+1+\rho)}{(2v+\rho)(2u+\rho)(v+u+\rho)}\,\hat{B}^{(k)}_b(v)\,R^{(k+1,k+1)}_{ab}(2v+\rho)\,P^{(k+1,k+1)}_{ab}\,\hat{a}^{(k)}(u)\\ &+ \frac{2v-1+\rho}{(v-u)(2v+\rho)}\,\hat{B}^{(k)}_b(v)\,R^{(k+1,k+1)}_{ab}(2v+\rho)\,\hat{D}^{(k+1)}_a(u)\,P^{(k+1,k+1)}_{ab},\end{aligned} \tag{3.1.21}$$

$$\begin{aligned}\hat{a}^{(k)}(v)\,\hat{D}^{(k+1)}_b(u) \equiv{}& \hat{D}^{(k+1)}_b(u)\,\hat{a}^{(k)}(v) + \frac{1}{v-u}\,\mathrm{tr}_a\,P^{(k+1,k+1)}_{ab}\Big(\hat{B}^{(k)}_b(u)\,\hat{C}^{(k)}_a(v) - \hat{B}^{(k)}_a(u)\,\hat{C}^{(k)}_b(u)\Big)\\ &+ \frac{1}{(v-u)(2u-1+\rho)}\,\mathrm{tr}_b\Big(\hat{B}^{(k)}_b(v)\,\hat{C}^{(k)}_b(u) - \hat{B}^{(k)}_b(u)\,\hat{C}^{(k)}_b(v)\Big)\cdot I^{(k+1,k+1)}_b.\end{aligned} \tag{3.1.22}$$

Proof. The $k=1$ case ($\mathcal{M}^{(1)} = \mathcal{M}$) is a restatement of the defining relations of $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$. When $k>1$ we additionally need to use Proposition 3.1.4. $\square$

3.1.3 Quantum spaces and monodromy matrices

Choose $m_1,\dots,m_{n-1} \in \mathbb{Z}_{\ge 0}$, the excitation numbers. Let $k = 1,\dots,n-1$. For each m_k, assign an m_k-tuple $\boldsymbol{u}^{(k)} = (u^{(k)}_1,\dots,u^{(k)}_{m_k})$ of complex parameters and a set of labels $\boldsymbol{a}^k = \{a^k_1,\dots,a^k_{m_k}\}$. We will use notation from [BR09] to denote multi-tuples:

$$\boldsymbol{u}^{(1\dots k)} := (\boldsymbol{u}^{(1)},\dots,\boldsymbol{u}^{(k)}), \qquad \boldsymbol{a}^{1\dots k} := (\boldsymbol{a}^1,\dots,\boldsymbol{a}^k). \tag{3.1.23}$$

We will say that M° is a *level-1 quantum space* and denote it by $M^{(1)}$. Then for each $2 \le k \le n$ we define a *level-k quantum space* $M^{(k)}$ recursively by

$$M^{(k)} := W^{(k)}_{\boldsymbol{a}^{k-1}} \otimes (M^{(k-1)})^0 \tag{3.1.24}$$

where

$$W^{(k)}_{\boldsymbol{a}^{k-1}} := V^{(k)}_{a^{k-1}_1} \otimes \cdots \otimes V^{(k)}_{a^{k-1}_{m_{k-1}}}$$

and $(M^{(k-1)})^0$ is level-$(k-1)$ *vacuum sector* defined by

$$(M^{(k-1)})^0 := \{\eta \in M^{(k-1)} : b^\circ_{ij}(u)\eta = 0 \text{ for } i > j \text{ and } j < k-1\}. \tag{3.1.25}$$

Propositions 3.1.3 and 3.1.4 imply that the space $M^{(k)}$ is a $\mathcal{B}^{\mathrm{ex}}_\rho(n-k+1, r-k+1)$- or $\mathcal{B}^{\mathrm{ex}}_\rho(n-k+1, 0)$-module for $k < r+1$ or $k \geq r+1$, respectively. In particular, for $k \geq 2$,

$$\begin{aligned} M^{(k)} = W^{(k)}_{\boldsymbol{a}^{k-1}} \otimes (\mathbb{C}e_1^{(k-1)})^{\otimes m_{k-2}} \otimes \cdots \otimes (\mathbb{C}e_1^{(2)})^{\otimes m_1} \\ \otimes L^{(k-1)}(\lambda^{(1)})_{c_1} \otimes \cdots \otimes L^{(k-1)}(\lambda^{(\ell)})_{c_\ell} \otimes V(\mu^\circ) \end{aligned}$$

where

$$L^{(k-1)}(\lambda^{(i)})_{c_i} := \{\eta \in L^\circ(\lambda^{(i)})_{c_i} : t^\circ_{ij}(u)\eta = 0 \text{ for } i > j \text{ and } j < k-1\}$$

are evaluation $Y(\mathfrak{gl}_{n-k+2})$-modules. (In the case when $L^\circ(\lambda^{(i)}) \cong \mathbb{C}^n$, i.e. the bulk quantum space is a tensor product of fundamental $\mathfrak{gl}_n$-modules, $L^{(k-1)}(\lambda^{(i)})_{c_i} \cong \mathbb{C}$.)

Definition 3.1.8. *We will say that $\hat{D}^{(1)}_a(v) := D^{(1)}_a(v)$ is a level-1 monodromy matrix. For each $2 \leq k \leq n$ we recursively define a level-k monodromy matrix, acting on the space $M^{(k)}$, via*

$$\begin{aligned} \hat{D}^{(k)}_{a\boldsymbol{a}^{1\ldots k-1}}\big(v; \boldsymbol{u}^{(1\ldots k-1)}\big) := \left(\prod_{i=1}^{m_{k-1}} R^{(k,k)}_{aa_i^{k-1}}\big(v + u_i^{(k-1)} + \rho\big) \right) \\ \times \hat{D}^{(k)}_{a\boldsymbol{a}^{1\ldots k-2}}\big(v; \boldsymbol{u}^{(1\ldots k-2)}\big) \left(\prod_{i=m_{k-1}}^{1} R^{(k,k)}_{aa_i^{k-1}}\big(v - u_i^{(k-1)}\big) \right), \end{aligned} \tag{3.1.26}$$

where $\hat{D}^{(k)}_{a\boldsymbol{a}^{1\ldots k-2}}(v; \boldsymbol{u}^{(1\ldots k-2)})$ is defined by (3.1.15),

$$\begin{aligned} \big[\hat{D}^{(k)}_{a\boldsymbol{a}^{1\ldots k-2}}\big(v; \boldsymbol{u}^{(1\ldots k-2)}\big)\big]_{ij} := \big[\hat{D}^{(k-1)}_{a\boldsymbol{a}^{1\ldots k-2}}\big(v + \tfrac{1}{2}; \boldsymbol{u}^{(1\ldots k-2)}\big)\big]_{1+i,1+j} \\ + \frac{\delta_{ij}}{2v+\rho}\big[\hat{D}^{(k-1)}_{a\boldsymbol{a}^{1\ldots k-2}}\big(v + \tfrac{1}{2}; \boldsymbol{u}^{(1\ldots k-2)}\big)\big]_{11} \end{aligned} \tag{3.1.27}$$

for $1 \leq i, j \leq n-k+1$.

The above definition outlines how the nested monodromy matrices will be constructed for this open spin chain, with extra R-matrices being attached at each level of nesting, and the extraction of appropriate submatrices. However, this recursive definition is unwieldy and we wish to instead have a closed form of the operator.

Proposition 3.1.9. *Let $\equiv$ denote equality of operators in the space $V^{(k)}_a \otimes M^{(k)}$ for any $2 \leq k \leq n$.*

Then

$$\hat{D}^{(k)}_{a\boldsymbol{a}^{1\ldots k-1}}\big(v;\boldsymbol{u}^{(1\ldots k-1)}\big) \equiv \left(\prod_{j=k-1}^{1}\prod_{i=1}^{m_j} R^{(k,j+1)}_{aa^j_i}\big(v+u^{(j)}_i+\tfrac{k-1-j}{2}+\rho\big)\right)$$
$$\times\, \hat{D}^{(k)}_a(v)\left(\prod_{j=1}^{k-1}\prod_{i=m_j}^{1} R^{(k,j+1)}_{aa^j_i}\big(v-u^{(j)}_i+\tfrac{k-1-j}{2}\big)\right). \tag{3.1.28}$$

We use the technical lemma below to help us prove Proposition 3.1.9. First, introduce a rational function

$$\Lambda^{\pm}(v;\boldsymbol{u}^{(k)}) := \prod_{i=1}^{m_k}\frac{(v+u^{(k)}_i\pm 1+\rho)(v-u^{(k)}_i\pm 1)}{(v+u^{(k)}_i+\rho)(v-u^{(k)}_i)}. \tag{3.1.29}$$

Lemma 3.1.10. *Let $A^{(k)}_a(v)\in\mathrm{End}(V^{(k)}_a)[[v^{-1}]]$ be a matrix operator such that $[A^{(k)}_a(v)]_{1+i,1}=0$ for $i\geq 1$. Set*

$$A^{(k)}_{a\boldsymbol{a}^{k-1}}(v;\boldsymbol{u}^{(k-1)}) := \left(\prod_{i=1}^{m_{k-1}} R^{(k,k)}_{aa^{k-1}_i}\big(v+u^{(k-1)}_i+\rho\big)\right)A^{(k)}_a(v)\left(\prod_{i=m_{k-1}}^{1} R^{(k,k)}_{aa^{k-1}_i}\big(v-u^{(k-1)}_i\big)\right).$$

Then, for $1\leq i,j\leq n-k$ and $\eta=(e^{(k)}_1)^{\otimes m_{k-1}}\in W^{(k)}_{\boldsymbol{a}^{k-1}}$,

$$\big[A^{(k)}_{a\boldsymbol{a}^{k-1}}(v;\boldsymbol{u}^{(k-1)})\big]_{11}\eta = \big[A^{(k)}_a(v)\big]_{11}\eta, \qquad \big[A^{(k)}_{a\boldsymbol{a}^{k-1}}(v;\boldsymbol{u}^{(k-1)})\big]_{1+i,1}\eta = 0, \tag{3.1.30}$$

$$\big[A^{(k)}_{a\boldsymbol{a}^{k-1}}(v;\boldsymbol{u}^{(k-1)})\big]_{1+i,1+j}\eta = \frac{1}{\Lambda^-(v;\boldsymbol{u}^{(k-1)})}\bigg(\big[A^{(k)}_a(v)\big]_{1+i,1+j}$$
$$+\,\delta_{ij}\frac{1-\Lambda^-(v;\boldsymbol{u}^{(k-1)})}{2v-1+\rho}\big[A^{(k)}_a(v)\big]_{11}\bigg)\eta. \tag{3.1.31}$$

Proof. The first two identities follow from

$$\big[R^{(k,k)}_{aa^{k-1}_l}(v)\big]_{11}\eta=\eta, \qquad \big[R^{(k,k)}_{aa^{k-1}_l}(v)\big]_{1+i,1}\eta=0.$$

To prove the third identity we need to use

$$\big[R^{(k,k)}_{aa^{k-1}_l}(v)\big]_{1+i,1+j}\eta = \frac{v}{v-1}\,\delta_{ij}\eta,$$

$$\big[R^{(k,k)}_{aa^{k-1}_l}(v)\big]_{1+i,1}\big[R^{(k,k)}_{aa^{k-1}_l}(u)\big]_{1,1+j}\eta = \frac{vu}{(v-1)(u-1)}\cdot\frac{1}{vu}\,\delta_{ij}\eta$$

giving

$$\big[M^{(k)}_{a\boldsymbol{a}^{k-1}}(v;\boldsymbol{u}^{(k-1)})\big]_{1+i,1+j}\eta = \frac{1}{\Lambda^-(v;\boldsymbol{u}^{(k-1)})}\Big(\big[M^{(k)}_a(v)\big]_{1+i,1+j}-\delta_{ij}f(v;\boldsymbol{u}^{(k-1)})\big[M^{(k)}_a(v)\big]_{11}\Big)\eta$$

where

$$f(v;\boldsymbol{u}^{(k-1)}) = \sum_{i=1}^{m_{k-1}} \frac{1}{(v+u_i^{(k-1)}+\rho)(v-u_i^{(k-1)})} \prod_{j=1}^{i-1} \frac{(v+u_i^{(k-1)}-1+\rho)(v-u_i^{(k-1)}-1)}{(v+u_i^{(k-1)}+\rho)(v-u_i^{(k-1)})}.$$

A simple induction on m_{k-1} then yields

$$f(v;\boldsymbol{u}^{(k-1)}) = \frac{1-\Lambda^{-}(v;\boldsymbol{u}^{(k-1)})}{2v-1+\rho},$$

implying the third identity. □

Proof of Proposition 3.1.9. It is sufficient to prove that (cf. (3.1.27))

$$\hat{D}^{(k)}_{a\boldsymbol{a}^{1\ldots k-2}}(v;\boldsymbol{u}^{(1\ldots k-2)}) \equiv \left(\prod_{j=k-2}^{1} \prod_{i=1}^{m_j} R^{(k,j+1)}_{aa_i^j}\big(v+u_i^{(j)}+\tfrac{k-1-j}{2}+\rho\big) \right) \times \hat{D}^{(k)}_{a}(v) \left(\prod_{j=1}^{k-2} \prod_{i=m_j}^{1} R^{(k,j+1)}_{aa_i^j}\big(v-u_i^{(j)}+\tfrac{k-1-j}{2}\big) \right). \tag{3.1.32}$$

We will use induction on k to prove the claim. The $k=2$ case follows from the definition and provides a base for induction. Now assume that the statement holds for $\hat{D}^{(k-1)}_{a\boldsymbol{a}^{1\ldots k-3}}(v;\boldsymbol{u}^{(1\ldots k-3)})$. Note that

$$\big[R^{(k,l)}_{ab}(v)\big]_{ij} e_1^{(l)} = \frac{v}{v-1}\,\delta_{ij} e_1^{(l)} \tag{3.1.33}$$

for $1 \le i,j \le n-k+1$ and any $k>l$. Combining this with Lemma 3.1.10 we obtain

$$\big[\hat{D}^{(k-1)}_{a\boldsymbol{a}^{1\ldots k-2}}(v+\tfrac12;\boldsymbol{u}^{(1\ldots k-2)})\big]_{11} \equiv \left(\prod_{l=k-3}^{1} \frac{1}{\Lambda^{-}\big(v+\frac{k-1-l}{2};\boldsymbol{u}^{(l)}\big)} \right) \big[\hat{D}^{(k-1)}_{a}(v+\tfrac12)\big]_{11},$$

$$\big[\hat{D}^{(k-1)}_{a\boldsymbol{a}^{1\ldots k-2}}(v+\tfrac12;\boldsymbol{u}^{(1\ldots k-2)})\big]_{1+i,1+j} \equiv \left(\prod_{l=k-2}^{1} \frac{1}{\Lambda^{-}\big(v+\frac{k-1-l}{2};\boldsymbol{u}^{(l)}\big)} \right) \left(\big[\hat{D}^{(k-1)}_{a}(v+\tfrac12)\big]_{1+i,1+j} + \delta_{ij}\,\frac{1-\Lambda^{-}\big(v+\frac12;\boldsymbol{u}^{(k-2)}\big)}{2v+\rho}\big[\hat{D}^{(k-1)}_{a}(v+\tfrac12)\big]_{11} \right)$$

for $1 \le i,j \le n-k+1$. The identities above together with (3.1.27) and (3.1.15) imply

$$\big[\hat{D}^{(k)}_{a\boldsymbol{a}^{1\ldots k-2}}\big(v;\boldsymbol{u}^{(1\ldots k-2)}\big)\big]_{ij} \equiv \left(\prod_{l=k-2}^{1} \frac{1}{\Lambda^{-}\big(v+\frac{k-1-l}{2};\boldsymbol{u}^{(l)}\big)} \right) \big[\hat{D}^{(k)}_{a}(v)\big]_{ij}$$

which is equivalent to (3.1.32), as required. □

The Corollary below follows from Propositions 3.1.9 and 3.1.4, and by virtue of the Yang-Baxter equation.

Corollary 3.1.11. *Let $\equiv$ denote equality of operators in the space $V_a^{(k)} \otimes V_b^{(k)} \otimes M^{(k)}$ for any $2 \le k \le n$. Then*

$$R_{ab}^{(k,k)}(v-w)\, D^{(k)}_{a\boldsymbol{a}^{1\dots k-1}}\big(v;\boldsymbol{u}^{(1\dots k-1)}\big)\, R_{ab}^{(k,k)}(v+w)\, D^{(k)}_{b\boldsymbol{a}^{1\dots k-1}}\big(w;\boldsymbol{u}^{(1\dots k-1)}\big)$$
$$\equiv D^{(k)}_{b\boldsymbol{a}^{1\dots k-1}}\big(w;\boldsymbol{u}^{(1\dots k-1)}\big)\, R_{ab}^{(k,k)}(v+w)\, D^{(k)}_{a\boldsymbol{a}^{1\dots k-1}}\big(v;\boldsymbol{u}^{(1\dots k-1)}\big)\, R_{ab}^{(k,k)}(v-w).$$

In other words, matrix entries of the level-k monodromy matrix satisfy the defining relations of the algebra $\widetilde{\mathcal{B}}_\rho(n-k+1, r-k+1)$ or $\widetilde{\mathcal{B}}_\rho(n-k+1,0)$ in $M^{(k)}$ for $r > k-1$ or $r \le k-1$, respectively.

3.1.4 Transfer matrix, creation operators and Bethe vectors

We are now ready to introduce transfer matrices and creation operators acting on the level-k quantum space $M^{(k)}$.

Definition 3.1.12. *The level-k a-operator is the first diagonal entry of the level-k monodromy matrix, namely*

$$\hat{a}^{(k)}_{\boldsymbol{a}^{1\dots k-1}}\big(v;\boldsymbol{u}^{(1\dots k-1)}\big) := \big[\hat{D}^{(k)}_{a\boldsymbol{a}^{1\dots k-1}}\big(v+\tfrac12;\boldsymbol{u}^{(1\dots k-1)}\big)\big]_{11}. \tag{3.1.34}$$

Definition 3.1.13. *The level-k transfer matrix for a $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$-chain is obtained by taking trace of the level-k monodromy matrix, namely*

$$\begin{aligned}\tau^{(k)}\big(v;\boldsymbol{u}^{(1\dots k-1)}\big) &:= \mathrm{tr}_a\, \hat{D}^{(k)}_{a\boldsymbol{a}^{1\dots k-1}}\big(v-\tfrac{k-1}{2};\boldsymbol{u}^{(1\dots k-1)}\big)\\ &= \frac{2v-n+\rho}{2v-k+\rho}\cdot \hat{a}^{(k)}_{\boldsymbol{a}^{1\dots k-1}}\big(v-\tfrac{k}{2};\boldsymbol{u}^{(1\dots k-1)}\big) + \mathrm{tr}_a\, \hat{D}^{(k+1)}_{a\boldsymbol{a}^{1\dots k-1}}\big(v-\tfrac{k}{2};\boldsymbol{u}^{(1\dots k-1)}\big).\end{aligned} \tag{3.1.35}$$

Our goal is to find eigenvectors (Bethe vectors) of the level-1 transfer matrix $\tau^{(1)}(v)$ and the corresponding eigenvalues. With this goal in mind we introduce a lowest weight vector with respect to the action of the level-n monodromy matrix,

$$\xi^{(n)} := (e_1^{(n)})^{\otimes m_{n-1}} \otimes \dots \otimes (e_1^{(2)})^{\otimes m_1} \otimes \xi \in M^{(n)}.$$

This vector will serve as a vacuum vector for constructing Bethe vectors of the full system, including auxiliary spaces of all nested systems.

Lemma 3.1.14. *The level-k a-operator acts on vector $\xi^{(n)}$ by*

$$\hat{a}^{(k)}_{\boldsymbol{a}^{1\dots k-1}}\big(v-\tfrac{k}{2};\boldsymbol{u}^{(1\dots k-1)}\big)\,\xi^{(n)} = \left(\prod_{i=1}^{k-2} \frac{1}{\Lambda^-\big(v-\tfrac{i}{2};\boldsymbol{u}^{(i)}\big)}\right) \frac{\widetilde{\gamma}^\circ_k(v)}{2v-k+1+\rho}\,\xi^{(n)}. \tag{3.1.36}$$

Proof. Recall that ξ is a lowest vector of weight $\gamma^\circ(v)$ with components $\gamma^\circ_i(v)$ determined by

(3.1.10). It follows from (3.1.13) and (3.2.18) that

$$\big[\hat{D}^{(k)}_a\big(v-\tfrac{k-1}{2}\big)\big]_{11}\xi^{(n)} = \Big(\gamma^\circ_k(v)+\sum_{i=1}^{k-1}\frac{\gamma^\circ_i(v)}{2v-k+1+\rho}\Big)\xi^{(n)} = \frac{\widetilde{\gamma}^\circ_k(v)}{2v-k+1+\rho}\,\xi^{(n)}.$$

All that remains is to apply Proposition 3.1.9, Lemma 3.1.10 and identity (3.1.33). $\square$

From now on we will view B-operators (cf. (3.1.12)) obtained from the nested monodromy matrix $\hat{D}^{(k)}_{a\boldsymbol{a}^{1\dots k}}\big(v;\boldsymbol{u}^{(1\dots k-1)}\big)$ as row-vectors, that is

$$\hat{B}^{(k)}_{a\boldsymbol{a}^{1\dots k-1}}\big(v;\boldsymbol{u}^{(1\dots k-1)}\big)\in(V_a^{(k+1)})^*\otimes\mathrm{End}(M^{(k)})[v^{-1}].$$

These row-vectors will give rise to level-k creation operators and level-k Bethe vectors. Since $M^{(k)}$ is a finite-dimensional vector space, we can evaluate the formal parameter v to any non-zero complex number.

Definition 3.1.15. *The level-k creation operator is defined by*

$$\mathscr{B}^{(k)}_{\boldsymbol{a}^{1\dots k}}\big(\boldsymbol{u}^{(1\dots k)}\big):=\prod_{i=1}^{m_k}\hat{B}^{(k)}_{a^k_i\boldsymbol{a}^{1\dots k-1}}\big(u^{(k)}_i;\boldsymbol{u}^{(1\dots k-1)}\big).$$

Note that operator $\mathscr{B}^{(k)}_{\boldsymbol{a}^{1\dots k}}\big(\boldsymbol{u}^{(1\dots k)}\big)$ is a row-vector with respect to all tensorands in $W^{(k+1)}_{\boldsymbol{a}^k}$.

Definition 3.1.16. *The level-k Bethe vector is defined by*

$$\Phi^{(k)}\big(\boldsymbol{u}^{(k\dots n-1)};\boldsymbol{u}^{(1,\dots k-1)}\big):=\prod_{i=k}^{n-1}\mathscr{B}^{(i)}_{\boldsymbol{a}^{1\dots i}}\big(\boldsymbol{u}^{(1\dots i)}\big)\cdot\xi^{(n)}.$$

where $\boldsymbol{u}^{(1,\dots k-1)}$ are viewed as fixed parameters.

The level-1 Bethe vector $\Phi^{(1)}\big(\boldsymbol{u}^{(1\dots n-1)}\big)\in M^{(1)}$ is a vector in the level-1 quantum space. For arbitrary $\boldsymbol{u}^{(1\dots n-1)}$ it is called an off-shell Bethe vector.

Set $\mathfrak{S}_{m_k,\dots,m_{n-1}}:=\mathfrak{S}_{m_k}\times\cdots\times\mathfrak{S}_{m_{n-1}}$. For any $\sigma^{(l)}\in\mathfrak{S}_{m_l}$ with $k\le l\le n-1$ define an action of $\mathfrak{S}_{m_k,\dots,m_{n-1}}$ on $\Phi^{(k)}(\boldsymbol{u}^{(k\dots n-1)})$ by

$$\sigma^{(l)}:\boldsymbol{u}^{(k\dots n-1)}\mapsto\boldsymbol{u}^{(k\dots n-1)}_{\sigma^{(l)}}:=(\boldsymbol{u}^{(k)},\dots,\boldsymbol{u}^{(l)}_{\sigma^{(l)}},\dots,\boldsymbol{u}^{(n-1)})$$

where we have set $\boldsymbol{u}^{(l)}_{\sigma^{(l)}}:=(u^{(l)}_{\sigma^{(l)}(1)},\dots,u^{(l)}_{\sigma^{(l)}(m_l)})$. The relation (3.1.19) together with the identity $\check{R}^{(l,l)}_{a^l_ia^l_j}(u)\,\xi^{(n)}=\xi^{(n)}$ implies the following Lemma.

Lemma 3.1.17. *The level-k Bethe vector $\Phi^{(k)}\big(\boldsymbol{u}^{(k\dots n-1)};\boldsymbol{u}^{(1\dots k-1)}\big)$ is invariant under the action of $\mathfrak{S}_{m_k,\dots,m_{n-1}}$.*

We are now ready to give the Bethe ansatz result for the spin chain.

Theorem 3.1.18. *The level-1 Bethe vector* $\Phi^{(1)}\big(\boldsymbol{u}^{(1\dots n-1)}\big)$ *is an eigenvector of* $\tau^{(1)}(v)$ *with the eigenvalue*

$$\begin{aligned}\Lambda^{(1)}\big(v;\boldsymbol{u}^{(1\dots n-1)}\big) := {} & \frac{2v-n+\rho}{2v-1+\rho}\,\Lambda^{+}\big(v-\tfrac12;\boldsymbol{u}^{(1)}\big)\,\frac{\widetilde{\gamma}^{\circ}_1(v)}{2v+\rho} + \Lambda^{-}\big(v-\tfrac{n-1}{2};\boldsymbol{u}^{(n-1)}\big)\,\frac{\widetilde{\gamma}^{\circ}_n(v)}{2v-n+1+\rho} \\ & + \sum_{i=2}^{n-1}\frac{2v-n+\rho}{2v-i+\rho}\,\Lambda^{-}\big(v-\tfrac{i-1}{2};\boldsymbol{u}^{(i-1)}\big)\,\Lambda^{+}\big(v-\tfrac{i}{2};\boldsymbol{u}^{(i)}\big)\,\frac{\widetilde{\gamma}^{\circ}_i(v)}{2v-i+1+\rho}\end{aligned} \tag{3.1.37}$$

provided

$$\operatorname*{Res}_{v\to u^{(j)}_i}\;\Lambda^{(1)}\big(v+\tfrac{j}{2};\boldsymbol{u}^{(1\dots n-1)}\big)=0 \tag{3.1.38}$$

for all $1\le i\le m_j$ *and* $1\le j\le n-1$.

Remark 3.1.19. The equations (3.1.38) are Bethe equations for a $\mathcal{B}^{\rm ex}_\rho(n,p)$-chain. Their explicit form is

$$\begin{aligned}&\frac{\widetilde{\gamma}^{\circ}_k\big(u^{(k)}_j+\frac{k}{2}\big)}{\widetilde{\gamma}^{\circ}_{k+1}\big(u^{(k)}_j+\frac{k}{2}\big)}\prod_{\substack{i=1\\ i\neq j}}^{m_k}\frac{(u^{(k)}_j-u^{(k)}_i+1)(u^{(k)}_j+u^{(k)}_i+1+\rho)}{(u^{(k)}_j-u^{(k)}_i-1)(u^{(k)}_j+u^{(k)}_i-1+\rho)} \\ &\quad=\prod_{i=1}^{m_{k-1}}\frac{(u^{(k)}_j-u^{(k-1)}_i+\frac12)(u^{(k)}_j+u^{(k-1)}_i+\frac12+\rho)}{(u^{(k)}_j-u^{(k-1)}_i-\frac12)(u^{(k)}_j+u^{(k-1)}_i-\frac12+\rho)} \\ &\qquad\times\prod_{i=1}^{m_{k+1}}\frac{(u^{(k)}_j-u^{(k+1)}_i+\frac12)(u^{(k)}_j+u^{(k+1)}_i+\frac12+\rho)}{(u^{(k)}_j-u^{(k+1)}_i-\frac12)(u^{(k)}_j+u^{(k+1)}_i-\frac12+\rho)}\end{aligned} \tag{3.1.39}$$

for $1\le j\le m_k$ and $1\le k\le n-1$ assuming $m_0=m_n=0$. For example, when $n=2$, we have $k=1$ and the r.h.s. of (3.1.39) equals 1.

Proof of Theorem 3.1.18. Using Lemma 3.1.7, symmetry $\mathfrak{S}_{m_k,\dots,m_{n-1}}$ of $\Phi^{(k)}\big(\boldsymbol{u}^{(k\dots n-1)};\boldsymbol{u}^{(1\dots k-1)}\big)$ and standard arguments, we obtain

$$\begin{aligned}&\hat{a}^{(k)}_{\boldsymbol{a}^{1\dots k-1}}\big(v-\tfrac{k}{2};\boldsymbol{u}^{(1\dots k-1)}\big)\,\Phi^{(k)}\big(\boldsymbol{u}^{(k\dots n-1)};\boldsymbol{u}^{(1\dots k-1)}\big) \\ &\quad=\bigg(\Lambda^{+}\big(v-\tfrac{k}{2};\boldsymbol{u}^{(k)}\big)\,\mathscr{B}^{(k)}_{\boldsymbol{a}^{1\dots k}}\big(\boldsymbol{u}^{(1\dots k)}\big)\,\hat{a}^{(k)}_{\boldsymbol{a}^{1\dots k-1}}\big(v-\tfrac{k}{2};\boldsymbol{u}^{(1\dots k-1)}\big) \\ &\qquad-\sum_{i=1}^{m_k}\frac{1}{v-\frac{k}{2}-u^{(k)}_i}\operatorname*{Res}_{w\to u^{(k)}_i}\Lambda^{+}\big(w;\boldsymbol{u}^{(k)}\big)\,\mathscr{B}^{(k)}_{\boldsymbol{a}^{1\dots k}}\big(\boldsymbol{u}^{(1\dots k)}_{\sigma^{(k)}_i,u^{(k)}_i\to v-\frac{k}{2}}\big)\,\hat{a}^{(k)}_{\boldsymbol{a}^{1\dots k-1}}\big(u^{(k)}_i;\boldsymbol{u}^{(1\dots k-1)}\big) \\ &\qquad-\sum_{i=1}^{m_k}\frac{2u^{(k)}_i+\rho}{(v-\frac{k}{2}+u^{(k)}_i+\rho)(2u^{(k)}_i-n+k+\rho)}\operatorname*{Res}_{w\to u^{(k)}_i}\Lambda^{-}\big(w;\boldsymbol{u}^{(k)}\big) \\ &\qquad\times\mathscr{B}^{(k)}_{\boldsymbol{a}^{1\dots k}}\big(\boldsymbol{u}^{(1\dots k)}_{\sigma^{(k)}_i,u^{(k)}_i\to v-\frac{k}{2}}\big)\,\mathrm{tr}_a\,\hat{D}^{(k+1)}_{a\boldsymbol{a}^{1\dots k}}\big(u^{(k)}_i;\boldsymbol{u}^{(1\dots k)}_{\sigma^{(k)}_i}\big)\bigg)\Phi^{(k+1)}\big(\boldsymbol{u}^{(k+1\dots n-1)};\boldsymbol{u}^{(1\dots k)}\big)\end{aligned} \tag{3.1.40}$$

and

$$
\begin{aligned}
&\operatorname{tr}_a \hat{D}^{(k+1)}_{a\boldsymbol{a}^{1\dots k-1}}\big(v-\tfrac{k}{2};\boldsymbol{u}^{(1\dots k-1)}\big)\,\Phi^{(k)}\big(\boldsymbol{u}^{(k\dots n-1)};\boldsymbol{u}^{(1\dots k-1)}\big)\\
&\quad=\Bigg(\Lambda^-\big(v-\tfrac{k}{2};\boldsymbol{u}^{(k)}\big)\,\mathscr{B}^{(k)}_{\boldsymbol{a}^{1\dots k}}\big(\boldsymbol{u}^{(1\dots k)}\big)\operatorname{tr}_a \hat{D}^{(k+1)}_{a\boldsymbol{a}^{1\dots k}}\big(v-\tfrac{k}{2};\boldsymbol{u}^{(1\dots k)}\big)\\
&\qquad-\sum_{i=1}^{m_1}\frac{2v-n+\rho}{(2v-k+\rho)(v-\frac{k}{2}+u_i^{(k)}+\rho)}\operatorname*{Res}_{w\to u_i^{(k)}}\Lambda^+\big(w;\boldsymbol{u}^{(k)}\big)\\
&\qquad\times\mathscr{B}^{(k)}_{\boldsymbol{a}^{1\dots k}}\big(\boldsymbol{u}^{(1\dots k)}_{\sigma_i^{(k)},u_i^{(k)}\to v-\frac{k}{2}}\big)\,\hat{\alpha}^{(k)}_{\boldsymbol{a}^{1\dots k-1}}\big(u_i^{(k)};\boldsymbol{u}^{(1\dots k-1)}\big)\\
&\qquad-\sum_{i=1}^{m_1}\frac{(2u_i^{(k)}+\rho)(2v-n+\rho)}{(2v-k+\rho)(v-\frac{k}{2}-u_i^{(k)})(2u_i^{(k)}-n+k+\rho)}\operatorname*{Res}_{w\to u_i^{(k)}}\Lambda^-\big(w;\boldsymbol{u}^{(k)}\big)\\
&\qquad\times\mathscr{B}^{(k)}_{\boldsymbol{a}^{1\dots k}}\big(\boldsymbol{u}^{(1\dots k)}_{\sigma_i^{(k)},u_i^{(k)}\to v-\frac{k}{2}}\big)\operatorname{tr}_a \hat{D}^{(k+1)}_{a\boldsymbol{a}^{1\dots k}}\big(u_i^{(k)};\boldsymbol{u}^{(1\dots k)}_{\sigma_i^{(k)}}\big)\Bigg)\Phi^{(k+1)}\big(\boldsymbol{u}^{(k+1\dots n-1)};\boldsymbol{u}^{(1\dots k)}\big).
\end{aligned}
\tag{3.1.41}
$$

Here $\sigma_i^{(k)}\in\mathfrak{S}_{m_k}$ denotes a cyclic permutation such that

$$
\boldsymbol{u}^{(k)}_{\sigma_i^{(k)}}=(u_i^{(k)},u_{i+1}^{(k)},\dots,u_{m_k}^{(k)},u_1^{(k)},u_2^{(k)},\dots,u_{i-1}^{(k)}).
$$

Below we indicate key identities that were used in obtaining (3.1.40) and (3.1.41). For this we need to introduce additional notation. Set $\boldsymbol{a}^k_{\not{\sigma}_j^k}=(a_1^k,\dots,a_{j-1}^k,a_{j+1}^k,\dots,a_{m_k}^k)$ and $\boldsymbol{u}^{(k)}_{\not{\sigma}_j^{(k)}}=(u_1^{(k)},\dots,u_{j-1}^{(k)},u_{j+1}^{(k)},\dots,u_{m_k}^{(k)})$. Then let

$$
\begin{aligned}
\hat{D}^{(k+1)}_{a\boldsymbol{a}^{1\dots k}_{\not{\sigma}_1^k}}\big(v;\boldsymbol{u}^{(1\dots k)}_{\sigma_j^{(k)},\not{\sigma}_j^{(k)}}\big)&:=\left(\prod_{i=2}^{m_k}R^{(k+1,k+1)}_{aa_i^k}\big(v+u^{(k)}_{\sigma_j^{(k)}(i)}+\rho\big)\right)\\
&\qquad\times\hat{D}^{(k+1)}_{a\boldsymbol{a}^{1\dots k-1}}\big(v;\boldsymbol{u}^{(1\dots k-1)}\big)\left(\prod_{i=m_k}^{2}R^{(k+1,k+1)}_{aa_i^k}\big(v-u^{(k)}_{\sigma_j^{(k)}(i)}\big)\right)
\end{aligned}
\tag{3.1.42}
$$

and

$$
\Lambda^{\pm}\big(w;\boldsymbol{u}^{(k)}_{\not{\sigma}_j^{(k)}}\big):=\prod_{\substack{i=1\\i\neq j}}^{m_k}\frac{(w-u_i^{(k)}\pm1)(w+u_i^{(k)}\pm1+\rho)}{(w-u_i^{(k)})(w+u_i^{(k)}+\rho)}
$$

so that

$$
\operatorname*{Res}_{w\to u_j^{(k)}}\Lambda^{\pm}(w;\boldsymbol{u}^{(k)})=\pm\frac{2u_j^{(k)}\pm1+\rho}{2u_j^{(k)}+\rho}\,\Lambda^{\pm}\big(u_j^{(k)};\boldsymbol{u}^{(k)}_{\not{\sigma}_j^{(k)}}\big).
$$

Also note that

$$
\operatorname{tr}_a R^{(k+1,k+1)}_{ab}(u)\,P^{(k+1,k+1)}_{ab}=\frac{u-n+k}{u-1}\cdot I^{(k+1,k+1)}_b.
$$

To obtain the second terms in the r.h.s. of (3.1.41) we used

$$\operatorname*{Res}_{w\to u_i^{(k)}} \Lambda^+\big(w;\boldsymbol{u}^{(k)}\big) = \frac{2u_i^{(k)}+1+\rho}{2u_i^{(k)}+\rho}\cdot\frac{2v-k-1+\rho}{2v-n+\rho}$$
$$\times\,\Lambda^+\big(u_i^{(k)};\boldsymbol{u}^{(k)}_{\not{u}_i^{(k)}}\big)\,\mathrm{tr}_a\,R^{(k+1,k+1)}_{aa_1^k}(2v-k+\rho)P^{(k+1,k+1)}_{aa_1^k}.$$

To obtain the third term in the r.h.s. of (3.1.40) we used the second equality below, and to obtain the third term in the r.h.s. of in (3.1.41) we used the third equality below:

$$\begin{aligned}
&\operatorname*{Res}_{w\to u_i^{(k)}} \Lambda^-\big(w;\boldsymbol{u}^{(k)}\big)\,\mathrm{tr}_a\,\hat D^{(k+1)}_{a\boldsymbol{a}^{1\dots k}}\big(u_i^{(k)};\boldsymbol{u}^{(1\dots k)}_{\sigma_i^{(k)}}\big)\\
&\quad= -\frac{2u_i^{(k)}-1+\rho}{2u_i^{(k)}+\rho}\Lambda^-\big(u_i^{(k)};\boldsymbol{u}^{(k)}_{\not{u}_i^{(k)}}\big)\,\mathrm{tr}_a\Big(R^{(k+1,k+1)}_{aa_1^k}\big(2u_i^{(k)}+\rho\big)\hat D^{(k+1)}_{a\boldsymbol{a}^{1\dots k}_{\not{a}_1^k}}\big(u_i^{(k)};\boldsymbol{u}^{(k)}_{\sigma_i^{(k)},\not{u}_i^{(k)}}\big)P^{(k+1,k+1)}_{aa_1^k}\Big)\\
&\quad= -\frac{2u_i^{(k)}-n+k+\rho}{2u_i^{(k)}+\rho}\Lambda^-\big(u_i^{(k)};\boldsymbol{u}^{(k)}_{\not{u}_i^{(k)}}\big)\hat D^{(k+1)}_{a_1^k\boldsymbol{a}^{1\dots k}_{\not{a}_1^k}}\big(u_i^{(k)};\boldsymbol{u}^{(k)}_{\sigma_i^{(k)},\not{u}_i^{(k)}}\big)\\
&\quad= -\frac{2u_i^{(k)}-n+k+\rho}{2u_i^{(k)}+\rho}\cdot\frac{2v-k-1+\rho}{2v-n+\rho}\\
&\qquad\times\Lambda^-\big(u_i^{(k)};\boldsymbol{u}^{(k)}_{\not{u}_i^{(k)}}\big)\,\mathrm{tr}_a\Big(R^{(k+1,k+1)}_{aa_1^k}(2v-k+\rho)\hat D^{(k+1)}_{a\boldsymbol{a}^{1\dots k}_{\not{a}_1^k}}\big(u_i^{(k)};\boldsymbol{u}^{(k)}_{\sigma_i^{(k)},\not{u}_i^{(k)}}\big)P^{(k+1,k+1)}_{aa_1^k}\Big).
\end{aligned}$$

Combining (3.1.40) and (3.1.41) gives

$$\begin{aligned}
&\tau^{(k)}\big(v;\boldsymbol{u}^{(1\dots k-1)}\big)\,\Phi^{(k)}\big(\boldsymbol{u}^{(k\dots n-1)};\boldsymbol{u}^{(1\dots k-1)}\big)\\
&\quad= \mathscr{B}^{(k)}_{\boldsymbol{a}^{1\dots k}}\big(\boldsymbol{u}^{(1\dots k)}\big)\bigg(\frac{2v-n+\rho}{2v-k+\rho}\,\Lambda^+\big(v-\tfrac{k}{2};\boldsymbol{u}^{(k)}\big)\,\hat{\mathscr{a}}^{(k)}_{\boldsymbol{a}^{1\dots k-1}}\big(v-\tfrac{k}{2};\boldsymbol{u}^{(1\dots k-1)}\big)\\
&\qquad\qquad+\Lambda^-\big(v-\tfrac{k}{2};\boldsymbol{u}^{(k)}\big)\,\mathrm{tr}_a\,\hat D^{(k+1)}_{a\boldsymbol{a}^{1\dots k}}\big(v-\tfrac{k}{2};\boldsymbol{u}^{(1\dots k)}\big)\bigg)\Phi^{(k+1)}\big(\boldsymbol{u}^{(k+1\dots n-1)};\boldsymbol{u}^{(1\dots k)}\big)\\
&\quad-\sum_{i=1}^{m_k}F_{n,k}\big(v,u_i^{(k)}\big)\,\mathscr{B}^{(k)}_{\boldsymbol{a}^{1\dots k}}\big(\boldsymbol{u}^{(1\dots k)}_{\sigma_i^{(k)},u_i^{(k)}\to v-\frac{k}{2}}\big)\\
&\qquad\times\operatorname*{Res}_{w\to u_i^{(k)}}\bigg(\frac{2w-n+k+\rho}{2w+\rho}\,\Lambda^+(w;\boldsymbol{u}^{(k)})\,\hat{\mathscr{a}}^{(k)}_{\boldsymbol{a}^{1\dots k-1}}\big(w;\boldsymbol{u}^{(1\dots k-1)}\big)\\
&\qquad\qquad+\Lambda^-(w;\boldsymbol{u}^{(k)})\,\mathrm{tr}_a\,\hat D^{(k+1)}_{a\boldsymbol{a}^{1\dots k}}\big(w;\boldsymbol{u}^{(1\dots k)}_{\sigma_i^{(k)}}\big)\bigg)\Phi^{(k+1)}\big(\boldsymbol{u}^{(k+1\dots n-1)};\boldsymbol{u}^{(1\dots k)}\big)
\end{aligned}\tag{3.1.43}$$

where

$$F_{n,k}(v,u)=\frac{(2v-n+\rho)(2u+\rho)}{(v-\frac{k}{2}-u)(v-\frac{k}{2}+u+\rho)(2u-n+k+\rho)}.$$

When $k=n-1$ and $n=2$, using (3.1.34) and (3.1.36) we have that $\Phi^{(k+1)}\big(\boldsymbol{u}^{(k+1\dots n-1)}\big)=\xi$ and

$$\hat{\mathscr{a}}^{(1)}(w)\,\xi=\frac{\widetilde{\gamma}^{\circ}_1(w+\frac12)}{2w+1+\rho}\,\xi,\qquad \mathrm{tr}_a\,\hat D^{(2)}_{a\boldsymbol{a}^1}\big(w;\boldsymbol{u}^{(1)}_{\sigma_i^{(1)}}\big)\,\xi=\frac{\widetilde{\gamma}^{\circ}_2(w+\frac12)}{2w+\rho}\,\xi,$$

yielding

$$
\begin{aligned}
&\tau^{(1)}(v)\,\Phi^{(1)}\big(\boldsymbol{u}^{(1)}\big) \\
&\quad= \bigg(\frac{2v-2+\rho}{2v-1+\rho}\,\Lambda^{+}\big(v-\tfrac12;\boldsymbol{u}^{(1)}\big)\,\frac{\widetilde\gamma^{\circ}_1(v)}{2v+\rho} + \Lambda^{-}\big(v-\tfrac12;\boldsymbol{u}^{(1)}\big)\frac{\widetilde\gamma^{\circ}_2(v)}{2v-1+\rho}\bigg)\Phi^{(1)}\big(\boldsymbol{u}^{(1)}\big) \\
&\qquad - \sum_{i=1}^{m_1} F_{2,1}\big(v,u^{(1)}_i\big)\,\operatorname*{Res}_{w\to u^{(1)}_i}\bigg(\frac{2w-1+\rho}{2w+\rho}\,\Lambda^{+}(w;\boldsymbol{u}^{(1)})\,\frac{\widetilde\gamma^{\circ}_1(w+\frac12)}{2w+1+\rho} \\
&\qquad\qquad\qquad\qquad + \Lambda^{-}(w;\boldsymbol{u}^{(1)})\,\frac{\widetilde\gamma^{\circ}_2(w+\frac12)}{2w+\rho}\bigg)\Phi^{(1)}\big(\boldsymbol{u}^{(1)}_{\sigma^{(1)}_i,u^{(1)}_i\to v-\frac12}\big) \\
&\quad= \Lambda^{(1)}\big(v-\tfrac12;\boldsymbol{u}^{(1)}\big)\,\Phi^{(1)}\big(\boldsymbol{u}^{(1)}\big) - \sum_{i=1}^{m_1} F_{2,1}\big(v,u^{(1)}_i\big)\operatorname*{Res}_{w\to u^{(1)}_i}\Lambda^{(1)}\big(w;\boldsymbol{u}^{(1)}\big)\,\Phi^{(1)}\big(\boldsymbol{u}^{(1)}_{\sigma^{(1)}_i,u^{(1)}_i\to v-\frac12}\big).
\end{aligned}
$$

This completes the proof when $n=2$. Assuming $1<k<n$ and $n>2$ introduce notation

$$
\begin{aligned}
\Lambda^{(k)}\big(v;\boldsymbol{u}^{(k-1\dots n-1)}\big) := \sum_{l=k}^{n-1}\frac{2v-n+\rho}{2v-l+\rho}\,\Lambda^{-}\big(v-\tfrac{l-1}{2};\boldsymbol{u}^{(l-1)}\big)\,\Lambda^{+}\big(v-\tfrac{l}{2};\boldsymbol{u}^{(l)}\big)\,\frac{\widetilde\gamma^{\circ}_l(v)}{2v-l+1+\rho} \\
+ \Lambda^{-}\big(v-\tfrac{n-1}{2};\boldsymbol{u}^{(n-1)}\big)\,\frac{\widetilde\gamma^{\circ}_n(v)}{2v-n+1+\rho}
\end{aligned}
$$

and notice that, for all $k\le l\le n-1$ and $1\le i\le m_l$,

$$
\operatorname*{Res}_{w\to u^{(l)}_i}\Lambda^{(k)}\big(w+\tfrac{k-1}{2};\boldsymbol{u}^{(k-1,k)}\big) = \operatorname*{Res}_{w\to u^{(l)}_i}\Lambda^{(1)}\big(w;\boldsymbol{u}^{(1\dots n-1)}\big).
$$

Hence, when $k=n-1$ and $n>2$, using similar arguments as before and symmetry of the Bethe vector, we find that

$$\begin{aligned}
&\tau^{(n-1)}\big(v;\boldsymbol{u}^{(1\dots n-2)}\big)\,\Phi^{(n-1)}\big(\boldsymbol{u}^{(n-1)};\boldsymbol{u}^{(1\dots n-2)}\big)\\
&\quad=\left(\prod_{j=1}^{n-2}\Lambda^{-}\big(v-\tfrac{j}{2};\boldsymbol{u}^{(j)}\big)\right)^{-1}\Bigg(\frac{2v-n+\rho}{2v-n+1+\rho}\,\Lambda^{-}\big(v-\tfrac{n-2}{2};\boldsymbol{u}^{(n-2)}\big)\\
&\qquad\qquad\times\Lambda^{+}\big(v-\tfrac{n-1}{2};\boldsymbol{u}^{(n-1)}\big)\,\frac{\widetilde{\gamma}^{\circ}_{n-1}(v)}{2v-n+2+\rho}\\
&\qquad\qquad+\Lambda^{-}\big(v-\tfrac{n-1}{2};\boldsymbol{u}^{(n-1)}\big)\,\frac{\widetilde{\gamma}^{\circ}_{n}(v)}{2v-n+1+\rho}\Bigg)\,\Phi^{(n-1)}\big(\boldsymbol{u}^{(n-1)};\boldsymbol{u}^{(1\dots n-2)}\big)\\
&\qquad-\sum_{i=1}^{m_{n-1}}F_{n,n-1}\big(v,u_i^{(n-1)}\big)\left(\prod_{j=1}^{n-2}\Lambda^{-}\big(u_i^{(n-1)}-\tfrac{j-n+1}{2};\boldsymbol{u}^{(j)}\big)\right)^{-1}\\
&\qquad\times\operatorname*{Res}_{w\to u_i^{(n-1)}}\Bigg(\frac{2w-1+\rho}{2w+\rho}\,\Lambda^{-}\big(w+\tfrac{1}{2};\boldsymbol{u}^{(n-2)}\big)\,\Lambda^{+}\big(w;\boldsymbol{u}^{(n-1)}\big)\,\frac{\widetilde{\gamma}^{\circ}_{n-1}(w+\frac{n-1}{2})}{2w+1+\rho}\\
&\qquad\qquad+\Lambda^{-}\big(w;\boldsymbol{u}^{(n-1)}\big)\,\frac{\widetilde{\gamma}^{\circ}_{n}(w+\frac{n-1}{2})}{2w+\rho}\Bigg)\,\Phi^{(n-1)}\big(\boldsymbol{u}^{(n-1)}_{u_i^{(n-1)}\to v-\frac{n-1}{2}};\boldsymbol{u}^{(1\dots n-2)}\big)\\
&\quad=\left(\prod_{j=1}^{n-2}\Lambda^{-}\big(v-\tfrac{j}{2};\boldsymbol{u}^{(j)}\big)\right)^{-1}\Lambda^{(n-1)}\big(v;\boldsymbol{u}^{(n-2,n-1)}\big)\,\Phi^{(n-1)}\big(\boldsymbol{u}^{(n-1)};\boldsymbol{u}^{(1\dots n-2)}\big)
\end{aligned}$$

provided $\operatorname*{Res}_{w\to u_i^{(n-1)}}\Lambda^{(1)}\big(w;\boldsymbol{u}^{(1\dots n-1)}\big)=0$ for all $1\le i\le m_{n-1}$. Next, when $1<k<n-1$ and $n>3$, using negative inductive arguments we obtain

$$\begin{aligned}
&\tau^{(k)}\big(v;\boldsymbol{u}^{(1\dots k-1)}\big)\,\Phi^{(k)}\big(\boldsymbol{u}^{(k\dots n-1)};\boldsymbol{u}^{(1\dots k-1)}\big)\\
&\qquad=\left(\prod_{j=1}^{k-1}\Lambda^{-}\big(v-\tfrac{j}{2};\boldsymbol{u}^{(j)}\big)\right)^{-1}\Lambda^{(k)}\big(v;\boldsymbol{u}^{(k-1\dots n-1)}\big)\,\Phi^{(k)}\big(\boldsymbol{u}^{(k\dots n-1)};\boldsymbol{u}^{(1\dots k-1)}\big)
\end{aligned}$$

provided $\operatorname*{Res}_{w\to u_i^{(l)}}\Lambda^{(1)}\big(w+\tfrac{l}{2};\boldsymbol{u}^{(1\dots n-1)}\big)=0$ for all $1\le i\le m_l$ and $k\le l\le n-1$. Finally, when $k=1$ and $n>2$, we obtain

$$\tau^{(1)}(v)\,\Phi^{(1)}\big(\boldsymbol{u}^{(1\dots n-1)}\big)=\Lambda^{(1)}\big(v;\boldsymbol{u}^{1\dots n-1}\big)\,\Phi^{(1)}\big(\boldsymbol{u}^{(1\dots n-1)}\big)$$

provided $\operatorname*{Res}_{w\to u_i^{(l)}}\Lambda^{(1)}\big(w+\tfrac{l}{2};\boldsymbol{u}^{(1\dots n-1)}\big)=0$ for all $1\le i\le m_l$ and $1\le l\le n-1$, which completes the proof. □

3.2 Nested algebraic Bethe ansatz for the $X_\rho(\mathfrak{g}_{2n},\mathfrak{g}_{2n}^{\theta})^{tw}$ spin chain

In this section we present the nested algebraic Bethe ansatz for the $X_\rho(\mathfrak{g}_{2n},\mathfrak{g}_{2n}^{\theta})^{tw}$ spin chain. We begin by reviewing the fusion procedure for $X(\mathfrak{g}_{2n})$, which allows us to construct representations of $X(\mathfrak{g}_{2n})$, which will form the bulk of the chain.

3.2.1 Representations of the Yangian $X(\mathfrak{g}_{2n})$

Recall the extended Yangian $X(\mathfrak{g}_{2n})$ from Section 1.2.2, generated by the matrix $T(u) \in \mathrm{End}(\mathbb{C}^{2n}) \otimes X(\mathfrak{g}_{2n})[[u^{-1}]]$ satisfying the RTT relation

$$R_{ab}(u-v)T_a(u)T_b(v) = T_b(v)T_a(u)R_{ab}(u-v), \tag{3.2.1}$$

with R-matrix

$$R(u) := I - \frac{P}{u} - \frac{Q}{\kappa - u}. \tag{3.2.2}$$

Here P is the permutation operator on $(\mathbb{C}^{2n})^{\otimes 2}$, $Q = P^{t_1}$ as defined in (1.2.22), and $\kappa = n \mp 1$ with the upper and lower signs denoting the orthogonal and symplectic cases respectively.

Since $X(\mathfrak{g}_{2n})$ cannot have an equivalent of the evaluation homomorphism, not every representation of $U(\mathfrak{g}_{2n})$ may be extended to a representation of $X(\mathfrak{g}_{2n})$. Nevertheless, we may extend some representations via a process known as *R-matrix fusion*.

The fusion procedure first makes use of the fact that the R-matrix itself satisfies the defining relation of $X(\mathfrak{g}_{2n})$ as a consequence of the Yang-Baxter relation, and therefore defines a representation of the algebra on $\mathbb{C}^{2n}$. This is the vector representation of $\mathfrak{g}_{2n}$ on $\mathbb{C}^{2n}$, a highest weight representation of weight $\lambda = (1, 0, \ldots, 0)$ and highest weight vector e_1, defined by the assignment $F_{ij} \mapsto e_{ij} - \theta_{ij}\, e_{2n-j+1,2n-i+1}$. The map

$$\varrho \;:\; t_{ij}(u) \mapsto \delta_{ij} + \frac{1}{u} e_{ij} - \frac{1}{u+\kappa} \theta_{ij}\, e_{2n-j+1,2n-i+1}$$

then equips $\mathbb{C}^{2n}$ with a structure of a $X(\mathfrak{g}_{2n})$-module. Since we will use lowest weight $X(\mathfrak{g}_{2n})$-modules, we compose the map ϱ with the anti-automorphisms *sign* and *tran*. We also include the shift automorphism τ_c as it will be necessary for the fusion procedure. Denoting the resulting map by $\boldsymbol{\varrho}_c := \varrho \circ sign \circ tran \circ \tau_c$ we have

$$\boldsymbol{\varrho}_c \;:\; t_{ij}(u) \mapsto \delta_{ij} - \frac{1}{u-c} e_{ji} + \frac{1}{u-c-\kappa} \theta_{ij}\, e_{2n-i+1,2n-j+1}.$$

It follows that

$$\boldsymbol{\varrho}_c(T(u)) = R(u-c), \qquad \boldsymbol{\varrho}_c(T(u))\, \boldsymbol{\varrho}_{-c}(T(-u)) = \boldsymbol{\varrho}_c(T(u))\, \boldsymbol{\varrho}_c(T^t(u+\kappa)) = 1 - \frac{1}{(u-c)^2}.$$

This allows us to view the space $\mathbb{C}^{2n}$ as an irreducible lowest weight $X(\mathfrak{g}_{2n})$-module with weight $\lambda(u)$ given by

$$\lambda_1(u) = 1 - \frac{1}{u-c}, \qquad \lambda_2(u) = \ldots = \lambda_{2n-1}(u) = 1, \qquad \lambda_{2n}(u) = 1 + \frac{1}{u-c-\kappa}. \tag{3.2.3}$$

We denote this module by $L(\lambda)_c$. We will use this notation for all irreducible finite-dimensional representations of $\mathfrak{g}_{2n}$ that can be equipped with a structure of a $X(\mathfrak{g}_{2n})$-module.

Consider the tensor product space $(\mathbb{C}^{2n})^{\otimes k}$ with $k \geq 2$. Each $\mathbb{C}^{2n}$ carries the vector representation of $\mathfrak{g}_{2n}$, and so the full vector space $(\mathbb{C}^{2n})^{\otimes k}$ is also a representation of $\mathfrak{g}_{2n}$. The Brauer algebra $\mathfrak{B}_k(\pm 2n)$ acts naturally on this tensor space and commutes with the action of $\mathfrak{g}_{2n}$, see e.g. Chapter 10 of [GW09]. The Brauer-Schur-Weyl duality allows us to obtain irreducible representations of $\mathfrak{g}_{2n}$ by studying primitive idempotents in $\mathfrak{B}_k(\pm 2n)$. Recall that irreducible representations of $\mathfrak{B}_k(\pm 2n)$ are labelled by all partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ of the non-negative integers k, $k-2$, $k-4$, ... Denote by λ' the partition conjugate to λ, e.g. if $\lambda = (2,1,1)$, then $\lambda' = (3,1)$. Then the vector space $(\mathbb{C}^{2n})^{\otimes k}$ decomposes as

$$(\mathbb{C}^{2n})^{\otimes k} \cong \bigoplus_{f=0}^{\lfloor k/2 \rfloor} \bigoplus_{\substack{\lambda \vdash k-2f \\ \lambda'_1 + \lambda'_2 \leq 2n}} V_\lambda \otimes L(\lambda)$$

in the orthogonal case, and as

$$(\mathbb{C}^{2n})^{\otimes k} \cong \bigoplus_{f=0}^{\lfloor k/2 \rfloor} \bigoplus_{\substack{\lambda \vdash k-2f \\ 2\lambda'_1 \leq 2n}} V_{\lambda'} \otimes L(\lambda)$$

in the symplectic case; here V_λ and $L(\lambda)$ are irreducible representations of $\mathfrak{B}_k(\pm 2n)$ and $\mathfrak{g}_{2n}$, respectively, labelled by the partition λ. We will focus on the symmetric representation labelled by the partition (k) and the skew-symmetric representation labelled by the partition $(1, \ldots, 1)$ of k. Assume that $k \geq 1$ in the orthogonal case and $1 \leq k \leq n$ in the symplectic case. By Theorem 2.2 of [IMO12] (see also Example 2.4 (iii) and Section 4 therein) the corresponding primitive idempotents act on the space $(\mathbb{C}^{2n})^{\otimes k}$ via operators $\Pi_k^\pm$ defined by

$$\Pi_1^\pm = 1 \quad \text{and} \quad \Pi_k^\pm = \frac{1}{k!} \prod_{i=2}^{k} \Big(R_{1i}(\mp(i-1)) \cdots R_{i-1,i}(\mp 1) \Big) \quad \text{if} \quad k \geq 2. \tag{3.2.4}$$

The subspace $L_k^\pm = \Pi_k^\pm (\mathbb{C}^{2n})^{\otimes k}$ is a $\mathfrak{g}_{2n}$-submodule of $(\mathbb{C}^{2n})^{\otimes k}$ that is isomorphic to the highest weight representation $L(\lambda)$ of weight $\lambda = (k, 0, \ldots, 0)$ in the orthogonal case and of weight $\lambda = (1, \ldots, 1, 0, \ldots, 0)$, where the number of 1's is k, in the symplectic case. The highest vector in the orthogonal case is

$$\eta = e_1 \otimes \cdots \otimes e_1.$$

In the symplectic case it is

$$\eta = \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sign}(\sigma)\, e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(k)},$$

where $\mathfrak{S}_k$ is the group of permutations on the set $\{1, 2, \ldots, k\}$.

By combining the comultiplication in (1.2.28) with the map ϱ_c and an appropriate choice of the shift automorphisms, we obtain a representation of $X(\mathfrak{g}_{2n})$ on the vector space $(\mathbb{C}^{2n})^{\otimes k}$ given by the assignment

$$T(u) \mapsto R_{01}(u-c)\, R_{02}(u-c\mp 1) \cdots R_{0k}(u-c\mp k\pm 1) \in \mathrm{End}((\mathbb{C}^{2n})^{\otimes (k+1)}) \tag{3.2.5}$$

where the "zero" space denotes the matrix space of $T(u)$.

Proposition 3.2.1. *The subspace $L_k^\pm \subset (\mathbb{C}^{2n})^{\otimes k}$ is stable under the action of $X(\mathfrak{g}_{2n})$ defined by (3.2.5). Moreover, the representation of $X(\mathfrak{g}_{2n})$ on $L_k^\pm$ obtained by restriction is an irreducible lowest weight representation of weight $\lambda(u)$ given by, for $1 \le i \le n$,*

$$\lambda_i(u) = 1 - \frac{\lambda_i}{u-c}, \qquad \lambda_{2n-i+1}(u) = 1 + \frac{\lambda_i}{u-c\mp k\pm 1-\kappa}, \tag{3.2.6}$$

where $\lambda = (k, 0, \ldots, 0)$ in the orthogonal case and $\lambda = (1, \ldots, 1, 0, \ldots, 0)$, with the number of 1's being k, in the symplectic case.

Proof. Using the explicit form of the idempotent $\Pi_k^\pm$ and the Yang-Baxter equation multiple times we find

$$\begin{aligned} &R_{01}(u-c)\, R_{02}(u-c\mp 1) \cdots R_{0k}(u-c\mp k\pm 1)\, \Pi_k^\pm \\ &\quad = \Pi_k^\pm\, R_{0k}(u-c\mp k\pm 1) \cdots R_{02}(u-c\mp 1)\, R_{01}(u-c), \end{aligned}$$

which implies the first part of the proposition. Since $U(\mathfrak{g}_{2n}) \subset X(\mathfrak{g}_{2n})$ we have $X(\mathfrak{g}_{2n})(e_1 \otimes \cdots \otimes e_1) = L_k^\pm$. By Lemma 5.17 in [AMR06] adapted to lowest weight representations, the tensor product of lowest vectors $e_1 \otimes \cdots \otimes e_1$ is again a lowest vector of weight given by the product of the individual weights with respect to the action (3.2.5), namely $\prod_{j=0}^{k-1} \lambda_i(u \mp j)$, where $\lambda_i(u \mp j)$ are those given by (3.2.3). This implies the second part of the proposition for the orthogonal case. For the symplectic case we refer the reader to the proof of Theorem 5.16 in [AMR06]. □

These representations of $X(\mathfrak{g}_{2n})$ will be denoted by $L(\lambda)_c$. We define the Lax operator $\mathcal{L}(u)$ of $X(\mathfrak{g}_{2n})$ by $T(u) \cdot L(\lambda)_c = \mathcal{L}(u-c)\, L(\lambda)_c$. It will be useful to know that

$$\mathcal{L}(u)\, \mathcal{L}^t(u+\kappa) = \mathcal{L}^t(u+\kappa)\, \mathcal{L}(u) = \prod_{i=0}^{k-1} \frac{(u\mp i)^2 - 1}{(u\mp i)^2} \cdot I = \frac{u\pm 1}{u} \cdot \frac{u\mp k}{u\mp k\pm 1} \cdot I \tag{3.2.7}$$

which follows from the relations $R(u)\, R^t(\kappa+u) = R^t(\kappa+u)\, R(u) = (1-u^{-2})I$ and (3.2.5).

3.2.2 The twisted Yangian $X_\rho(\mathfrak{g}_{2n}, \mathfrak{g}_{2n}^\theta)^{tw}$

We now focus on the extended twisted Yangian $X_\rho(\mathfrak{g}_{2n}, \mathfrak{g}_{2n}^\theta)^{tw}$ and its representation theory, adhering closely to [GR16, GRW17, GRW19]. This will form the full underlying algebra of the open spin chain, and many results in this section will bear resemblance to the those given in Section 3.1.1 for the $\mathfrak{gl}_n$ chain. As in the $\mathcal{B}_\rho^{\mathrm{ex}}(n,p)$ case above, we have included an additional "shift" parameter ρ in the definition.

Recall the C and D symmetric pairs from Section 1.3.2. Each of these is defined using an involutive automorphism θ of the form $F \mapsto GFG^{-1}$. In order to construct the twisted Yangian, we use this matrix G to construct a specific matrix-valued rational function

$$G(u) = \frac{dI - uG}{d-u} \qquad \text{where} \qquad d = \frac{1}{4}\operatorname{tr} G, \tag{3.2.8}$$

so that $d = 0$ for symmetric pairs CI and DIII, $d = n/2$ for CD0, and $d = (p-q)/4$ for CII and DI. Note that in each case the matrix $G(u) = G + O(u^{-1})$ when expanded in powers of u^{-1}.

Definition 3.2.2. *The extended twisted Yangian $X_\rho(\mathfrak{g}_{2n}, \mathfrak{g}_{2n}^\theta)^{tw}$ is the subalgebra of $X(\mathfrak{g}_{2n})$ generated by the coefficients of the entries of the matrix*

$$\Sigma(u) = T(u - \tfrac{\kappa}{2})\, G(u + \tfrac{\rho}{2})\, T^t(\tilde{u} - \tfrac{\kappa}{2}) \qquad \textit{where} \qquad \tilde{u} = \kappa - u - \rho. \tag{3.2.9}$$

The "ρ-shifted" twisted Yangian defined above is isomorphic to the one introduced by one of the authors in [GR16]. The isomorphism is provided by the map $\Sigma(u) \mapsto S(u + \frac{\rho}{2})$. (Note that $\Sigma(u)$ is used to denote the special twisted Yangian in [GR16].)

Note that, from this definition, it is immediate that the extended twisted Yangian $X_\rho(\mathfrak{g}_{2n}, \mathfrak{g}_{2n}^\theta)^{tw}$ is indeed a coideal subalgebra of $X(\mathfrak{g}_{2n})$, and we have

$$\Delta\big(\sigma_{ij}(u)\big) = \sum_{lm} t_{il}(u - \tfrac{\kappa}{2})\, t_{2n-j+1,2n-m+1}(\tilde{u} - \tfrac{\kappa}{2}) \otimes \sigma_{lm}(u) \in X(\mathfrak{g}_{2n}) \otimes X_\rho(\mathfrak{g}_{2n}, \mathfrak{g}_{2n}^\theta)^{tw}.$$

The Lemma below gives the defining relations of the algebra, and is due to Lemmas 4.1 and 4.3 in [GR16].

Lemma 3.2.3. *The matrix $\Sigma(u)$ defined in (3.2.9) satisfies the reflection equation and the symmetry relation:*

$$R(u-v)\,\Sigma_1(u)\,R(u+v+\rho)\,\Sigma_2(v) = \Sigma_2(v)\,R(u+v+\rho)\,\Sigma_1(u)\,R(u-v), \tag{3.2.10}$$

$$\Sigma^t(u) = (\pm)\,\Sigma(\tilde{u}) \pm \frac{\Sigma(u) - \Sigma(\tilde{u})}{u - \tilde{u}} + \frac{\operatorname{tr}(G(u + \frac{\rho}{2}))\,\Sigma(\tilde{u}) - \operatorname{tr}(\Sigma(u)) \cdot I}{u - \tilde{u} - \kappa}, \tag{3.2.11}$$

where the lower sign in $(\pm)$ distinguishes symmetric pairs CI and DIII from the remaining ones.

The name extended twisted Yangian refers to the lack of unitarity relation of the algebra which, just as in the $\mathcal{B}_\rho^{\mathrm{ex}}(n,p)$ case, implies the existence of a nontrivial centre of the algebra. If the unitarity relation is included, one arrives at the *twisted Yangian* $Y(\mathfrak{g}_{2n}, \mathfrak{g}_{2n}^\theta)^{tw}$. Alternatively, one arrives at the twisted Yangian using the construction (3.2.9), but instead starting from the Yangian $Y(\mathfrak{g}_{2n})$ instead of its extended version; this is a consequence of the cross-unitarity relation. In the other direction, one may drop the symmetry relation to define an *extended reflection algebra*, which has the reflection equation as its only relation, just as we defined $\mathcal{B}_\rho^{\mathrm{ex}}(n,p)$. These algebras were studied in [GR16]. Since we will make use of the symmetry relation but not the unitarity relation in the nested algebraic Bethe ansatz, we have opted to use $X_\rho(\mathfrak{g}_{2n}, \mathfrak{g}_{2n}^\theta)^{tw}$ as our starting point.

Note above that the cases CI and DIII must be distinguished from the others using an additional $(\pm)$ sign in (3.2.11). We can combine these cases with the others by studying a slightly different generating matrix. First we introduce the rational function

$$g(u) = \begin{cases} 1 & \text{for CI, DIII,} \\ 2u - \kappa \pm 1 + \rho & \text{for CII, DI when } p = q, \\ \dfrac{u - \tilde{u} - \kappa}{\operatorname{tr}(G(u + \frac{\rho}{2}))} & \text{for CD0 and CII, DI when } p > q. \end{cases} \tag{3.2.12}$$

Note that in the last case we have

$$\frac{u - \tilde{u} - \kappa}{\operatorname{tr}(G(u + \frac{\rho}{2}))} = \frac{(u - \kappa + \frac{\rho}{2})(u - d + \frac{\rho}{2})}{d(2u - n + \rho)}.$$

Now we define the matrix

$$S(u) := g(u)\Sigma(u) \in X_\rho(\mathfrak{g}_{2n}, \mathfrak{g}_{2n}^\theta)^{tw}((u^{-1})). \tag{3.2.13}$$

Lemma 3.2.4. *The matrix $S(u)$ satisfies the reflection equation and the symmetry relation*

$$S^t(u) = -\left(1 \pm \frac{1}{u - \tilde{u}}\right) S(\tilde{u}) \pm \frac{S(u)}{u - \tilde{u}} - \frac{\operatorname{tr}(S(u)) \cdot I}{u - \tilde{u} - \kappa}. \tag{3.2.14}$$

Proof. Since we have only multiplied by a rational function, the reflection equation is satisfied as before.

Substituting (3.2.13) to (3.2.14) gives

$$\Sigma^t(u) = -\frac{g(\tilde{u})}{g(u)}\left(1 \pm \frac{1}{u - \tilde{u}}\right)\Sigma(\kappa - u - \rho) \pm \frac{\Sigma(u)}{u - \tilde{u}} - \frac{\operatorname{tr}(\Sigma(u)) \cdot I}{u - \tilde{u} - \kappa}. \tag{3.2.15}$$

For symmetric pairs CI and DIII we have $g(u) = 1$ giving

$$-\frac{g(\tilde{u})}{g(u)}\left(1 \pm \frac{1}{u - \tilde{u}}\right) = -1 \mp \frac{1}{u - \tilde{u}}.$$

For symmetric pairs CII and DI when $p = q$ we have instead $g(u) = 2u - \kappa + 1 + \rho$ and so

$$-\frac{g(\tilde{u})}{g(u)}\left(1 \mp \frac{1}{u - \tilde{u}}\right) = 1 \mp \frac{1}{u - \tilde{u}}.$$

Thus for the above symmetric pairs (3.2.15) becomes

$$\Sigma^t(u) = \left((\pm)1 \mp \frac{1}{u - \tilde{u}}\right)\Sigma(\kappa - u - \rho) \pm \frac{\Sigma(u)}{u - \tilde{u}} - \frac{\operatorname{tr}(\Sigma(u)) \cdot I}{u - \tilde{u} - \kappa},$$

which is equivalent to (3.2.11), since the above cases have $\mathrm{tr}(G(u))=0$.

Let us now focus on all the remaining symmetric pairs. By Lemma 2.2 in [GRW17] the matrix $G(u)$ itself satisfies the symmetry relation (3.2.11), namely

$$
\begin{aligned}
G^t(u+\tfrac{\rho}{2}) = G(\kappa-u-\tfrac{\rho}{2}) \pm \frac{G(u+\frac{\rho}{2})-G(\kappa-u-\frac{\rho}{2})}{u-\tilde{u}} \\
+ \frac{\mathrm{tr}(G(u+\frac{\rho}{2}))G(\kappa-u-\frac{\rho}{2})-\mathrm{tr}(G(u+\frac{\rho}{2}))\cdot I}{u-\tilde{u}-\kappa}.
\end{aligned}
$$

Recall (3.2.12). Taking the trace of both sides we find

$$
-\frac{u-\tilde{u}-\kappa}{2u+\rho}\left(1\mp\frac{1}{u-\tilde{u}}+\frac{2\kappa\pm 2}{u-\tilde{u}-\kappa}\right)g(\tilde{u}) = \left(1\mp\frac{1}{u-\tilde{u}}+\frac{g^{-1}(u)}{u-\tilde{u}-\kappa}\right)g(u)
$$

and rearrange to

$$
-\frac{g(\tilde{u})}{g(u)}\left(1\pm\frac{1}{u-\tilde{u}}\right) = \left(1\mp\frac{1}{u-\tilde{u}}+\frac{g^{-1}(u)}{u-\tilde{u}-\kappa}\right).
$$

This allows us to rewrite (3.2.15) as

$$
\begin{aligned}
\Sigma^t(u) &= \left(1\mp\frac{1}{u-\tilde{u}}+\frac{g^{-1}(u)}{u-\tilde{u}-\kappa}\right)\Sigma(\kappa-u-\rho)\pm\frac{\Sigma(u)}{u-\tilde{u}}-\frac{\mathrm{tr}(\Sigma(u))\cdot I}{u-\tilde{u}-\kappa} \\
&= \Sigma(\tilde{u})\pm\frac{\Sigma(u)-\Sigma(\tilde{u})}{u-\tilde{u}}+\frac{\mathrm{tr}(G(u+\frac{\rho}{2}))\Sigma(\tilde{u})-\mathrm{tr}(\Sigma(u))\cdot I}{u-\tilde{u}-\kappa},
\end{aligned}
$$

which coincides with the symmetry relation (3.2.11), as required. □

This new symmetry relation (3.2.14) is more convenient than (3.2.11) in the context of the nested algebraic Bethe ansatz for the $X_\rho(\mathfrak{g}_{2n},\mathfrak{g}_{2n}^\theta)^{tw}$-chain as it allows us to study all types of boundary at once. This will become evident in Sections 3.2.7 and 3.2.8, where the exchange relations are obtained.

Next, we introduce the lowest weight representations of $X_\rho(\mathfrak{g}_{2n},\mathfrak{g}_{2n}^\theta)^{tw}$. We rephrase some of the statements given in Section 4 of [GRW17], where the highest weight representation theory was studied.

Definition 3.2.5. *A representation V of $X_\rho(\mathfrak{g}_{2n},\mathfrak{g}_{2n}^\theta)^{tw}$ is called a lowest weight representation if there exists a non-zero vector $\xi\in V$ such that $V=X_\rho(\mathfrak{g}_{2n},\mathfrak{g}_{2n}^\theta)^{tw}\,\xi$ and*

$$
\sigma_{ij}(u)\xi=0 \quad \text{for} \quad 1\le j<i\le 2n \quad \text{and} \quad \sigma_{ii}(u)\xi=\mu_i(u)\xi \quad \text{for} \quad 1\le i\le n, \tag{3.2.16}
$$

where $\mu_i(u)$ are formal power series in u^{-1} with the constant term equal g_{ii}. The vector ξ is called the lowest weight vector of V, and the n-tuple $\mu(u)=(\mu_1(u),\dots,\mu_n(u))$ is called the lowest weight of V.

Note that the symmetry relation (3.2.11) implies that ξ is also an eigenvector for $\sigma_{ii}(u)$ with $n<i\le 2n$.

Our focus will be on the lowest weight $X_\rho(\mathfrak{g}_{2n},\mathfrak{g}_{2n}^\theta)^{tw}$-modules obtained by tensoring lowest weight $X(\mathfrak{g}_{2n})$- and $X_\rho(\mathfrak{g}_{2n},\mathfrak{g}_{2n}^\theta)^{tw}$-representations. With this goal in mind we make use of Proposition 4.10 in [GRW17], which is rephrased to include the shift ρ.

Proposition 3.2.6. *Let η be the lowest vector of a lowest weight $X(\mathfrak{g}_{2n})$-module $L(\lambda(u))$ and let ξ be the lowest vector of a lowest weight $X_\rho(\mathfrak{g}_{2n},\mathfrak{g}_{2n}^\theta)^{tw}$-module $V(\mu(u))$. Then $X_\rho(\mathfrak{g}_{2n},\mathfrak{g}_{2n}^\theta)^{tw}(\eta\otimes\xi)$ is a lowest weight $X_\rho(\mathfrak{g}_{2n},\mathfrak{g}_{2n}^\theta)^{tw}$-module with the lowest vector $\eta\otimes\xi$ and the lowest weight $\gamma(u)$ with components determined by the relations*

$$\widetilde{\gamma}_i(u)=\widetilde{\mu}_i(u)\,\lambda_i(u-\tfrac{\kappa}{2})\,\lambda_{2n-i+1}(\tilde{u}-\tfrac{\kappa}{2})\quad \textit{for}\quad 1\le i\le n, \tag{3.2.17}$$

with

$$\widetilde{\mu}_i(u):=(2u+\rho-i+1)\,\mu_i(u)+\sum_{j=1}^{i-1}\mu_j(u), \tag{3.2.18}$$

and $\widetilde{\gamma}_i(u)$ defined analogously.

We introduce the one-dimensional representations of $X_\rho(\mathfrak{g}_{2n},\mathfrak{g}_{2n}^\theta)^{tw}$ below, which play the role of boundary conditions for the spin chain. This Lemma rephrases Lemma 2.3 in [GRW17] and Lemma 5.4 in [GRW19].

Lemma 3.2.7. *Let $a,b\in\mathbb{C}$. Then the matrices*

$$K(u)=G-\frac{a}{u+\frac{\rho}{2}}I \tag{3.2.19}$$

when $n\ge 1$ and G is type CI, or $n\ge 2$ and G is of type DIII, and

$$\begin{aligned}K(u)=&-\left(1-\frac{b}{u+\frac{\rho}{2}}\right)\left(\left(1-\frac{a}{u+\frac{\rho}{2}}\right)e_{11}-\left(1+\frac{a}{u+\frac{\rho}{2}}\right)e_{22}\right)\\&+\left(1+\frac{b}{u+\frac{\rho}{2}}\right)\left(\left(1-\frac{a}{u+\frac{\rho}{2}}\right)e_{33}-\left(1+\frac{a}{u+\frac{\rho}{2}}\right)e_{44}\right),\end{aligned} \tag{3.2.20}$$

when $n=2$, and

$$K(u)=\frac{(u-a+\frac{\rho}{2})(u+a-2d+\frac{\rho}{2})}{(u-d+\frac{\rho}{2})^2}\left(I-\frac{2u+\rho}{u-a+\frac{\rho}{2}}e_{11}-\frac{2u+\rho}{u+a-2d+\frac{\rho}{2}}e_{2n,2n}\right), \tag{3.2.21}$$

when $n>2$ and $d=\frac{n}{2}-1$, are one- or two-parameter solutions of (3.2.10) satisfying the symmetry relation (3.2.11) (with $\Sigma(u)$ replaced by $K(u)$).

The non-zero matrix elements of $K(u)$ in (3.2.19-3.2.21) are power series in u^{-1} of the form $g_{ii}+u^{-1}\mathbb{C}[[u^{-1}]]$, so that $K(u)\in G+u^{-1}\mathbb{C}[[u^{-1}]]$ with G type DI with $p=2$ for (3.2.20–3.2.21). This implies the following statement.

Proposition 3.2.8. *(i) The assignment $\Sigma(u)\mapsto K(u)$ yields a one-dimensional representation of $X_\rho(\mathfrak{g}_{2n},\mathfrak{g}_{2n}^\theta)^{tw}$ of weight $\mu(u)$ given by, in the case-by-case way,*

- *for CI and DIII by* (3.2.19)*:*

$$\mu_1(u) = \ldots = \mu_n(u) = 1 - \frac{a}{u + \frac{\rho}{2}}, \tag{3.2.22}$$

- *for DI when $n = p = q = 2$ by* (3.2.20)*:*

$$\begin{aligned} \mu_1(u) &= \left(-1 + \frac{a}{u+\frac{\rho}{2}}\right)\left(1 - \frac{b}{u+\frac{\rho}{2}}\right), \\ \mu_2(u) &= \left(1 + \frac{a}{u+\frac{\rho}{2}}\right)\left(1 - \frac{b}{u+\frac{\rho}{2}}\right), \end{aligned} \tag{3.2.23}$$

- *for DI when $n > 2$, $p = 2n-2$, $q = 2$ by* (3.2.21)*:*

$$\begin{aligned} \mu_1(u) &= -\frac{(u+a+\frac{\rho}{2})(u+a-2d+\frac{\rho}{2})}{(u-d+\frac{\rho}{2})^2}, \\ \mu_2(u) = \ldots = \mu_n(u) &= \frac{(u-a+\frac{\rho}{2})(u+a-2d+\frac{\rho}{2})}{(u-d+\frac{\rho}{2})^2}. \end{aligned} \tag{3.2.24}$$

(ii) The assignment $\Sigma(u) \mapsto K(u) = G(u+\frac{\rho}{2})$ with $G(u)$ defined by (3.2.8) *yields a one-dimensional representation of $X_\rho(\mathfrak{g}_{2n}, \mathfrak{g}_{2n}^\theta)^{tw}$ of weight $\mu(u)$ given, case-by-case, by*

- *for CII when $p \geq q$ and DI when $p \geq q \geq 4$:*

$$\mu_i(u) = \frac{d - (u+\frac{\rho}{2})g_{ii}}{d - u - \frac{\rho}{2}} \quad \text{for} \quad 1 \leq i \leq n, \tag{3.2.25}$$

- *for CD0:*

$$\mu_1(u) = \ldots = \mu_n(u) = 1. \tag{3.2.26}$$

Finally, we introduce the spin chain model using the above representation theory defining the action of $X_\rho(\mathfrak{g}_{2n}, \mathfrak{g}_{2n}^\theta)^{tw}$ on it.

Recall the module $L(\lambda)_c$, regarded as a $X(\mathfrak{g}_{2n})$ lowest weight module, with action defined by the fusion procedure in Proposition 3.2.1. Let $V(\mu)$ denote a one-dimensional representation of $X_\rho(\mathfrak{g}_{2n}, \mathfrak{g}_{2n}^\theta)^{tw}$ of the sort given in Proposition 3.2.8. The spin chain is then

$$M := L(\lambda^{(1)})_{c_1} \otimes L(\lambda^{(2)})_{c_2} \otimes \ldots \otimes L(\lambda^{(\ell)})_{c_\ell} \otimes V(\mu), \tag{3.2.27}$$

The generating matrix $S(u)$ (cf. (3.2.13)) of $X_\rho(\mathfrak{g}_{2n}, \mathfrak{g}_{2n}^\theta)^{tw}$ acts on this space as

$$S(u) \cdot M = g(u)\left(\prod_{i=1}^{\ell} \mathcal{L}_i(u - c_i - \tfrac{\kappa}{2})\right) K(u) \left(\prod_{i=\ell}^{1} \mathcal{L}_i^t(\tilde{u} - c_i - \tfrac{\kappa}{2})\right) M, \tag{3.2.28}$$

where $\mathcal{L}_i(u)$ are the fused Lax operators of $X(\mathfrak{g}_{2n})$, $K(u)$ are given by Lemma 3.2.7, and $\tilde{u} = \kappa - u - \rho$. By Proposition 3.2.6, the space M is a lowest weight $X_\rho(\mathfrak{g}_{2n}, \mathfrak{g}_{2n}^\theta)^{tw}$-module of weight $\gamma(u)$ with components defined by (3.2.17) with $\mu_i(u)$ as in Proposition 3.2.8 and $\lambda_i(u)$ given by

$$\lambda_i(u) = \prod_{j=1}^{\ell} \lambda_i^{(j)}(u) \tag{3.2.29}$$

with weights $\lambda_i^{(j)}(u)$ as in (3.2.6). The image of $S(u)$ on M given by (3.2.28) is the *monodromy matrix* of the open spin chain.

3.2.3 Block decomposition of $X(\mathfrak{g}_{2n})$ and $X_\rho(\mathfrak{g}_{2n}, \mathfrak{g}_{2n}^\rho)^{tw}$

We now initiate the nested algebraic Bethe ansatz of the chain, using the nesting procedure from Chapter 2. The first step is to decompose the generating matrices $T(u)$ and $S(u)$ into $n \times n$ block matrices, and recast the defining relations in block form. We decompose the $2n \times 2n$ matrices $T(u)$ and $S(u)$ into $n \times n$ blocks as follows:

$$T(u) = \begin{pmatrix} \overline{A}(u) & \overline{B}(u) \\ \overline{C}(u) & \overline{D}(u) \end{pmatrix}, \qquad S(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{3.2.30}$$

We will denote the matrix elements of $A(u)$ by $a_{ij}(u)$ with $1 \le i, j \le n$, and similarly for matrices $B(u)$, $C(u)$ and $D(u)$, and their barred counterparts.

Recall that $\mathbb{C}^{2n} \cong \mathbb{C}^2 \otimes \mathbb{C}^n$. Let e_{ij} with $1 \le i, j \le 2n$ denote the standard matrix units of $\mathrm{End}(\mathbb{C}^{2n})$. Moreover, let x_{ij} with $1 \le i, j \le 2$ (resp. e_{ij} with $1 \le i, j \le n$) denote the standard matrix units of $\mathrm{End}(\mathbb{C}^2)$ (resp. $\mathrm{End}(\mathbb{C}^n)$). Then, for any $1 \le i, j \le n$, we may write

$$\mathsf{e}_{ij} = x_{11} \otimes e_{ij}, \quad \mathsf{e}_{n+i,j} = x_{21} \otimes e_{ij}, \quad \mathsf{e}_{i,n+j} = x_{12} \otimes e_{ij}, \quad \mathsf{e}_{n+i,n+j} = x_{22} \otimes e_{ij}. \tag{3.2.31}$$

Hence any matrix $M \in \mathrm{End}(\mathbb{C}^{2n})$ with entries $(M)_{ij} \in \mathbb{C}$ can be written as

$$M = \sum_{a,b=1}^{2} x_{ab} \otimes [\![M]\!]_{ab} \in \mathrm{End}(\mathbb{C}^2) \otimes \mathrm{End}(\mathbb{C}^n),$$

where $[\![M]\!]_{ab} = \sum_{i,j=1}^{n} [M]_{i+n(a-1),j+n(b-1)}\, e_{ij}$ are blocks of M, viz. (3.2.30). Now suppose that $M \in \mathrm{End}(\mathbb{C}^{2n} \otimes \mathbb{C}^{2n})$. Then we may write

$$M = \sum_{a,b,c,d=1}^{2} x_{ab} \otimes x_{cd} \otimes [\![M]\!]_{abcd} \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n),$$

where $[\![M]\!]_{abcd}$ are obtained as follows. Writing $M = \sum_{i,j,k,l=1}^{2n} [M]_{ijkl}\, \mathsf{e}_{ij} \otimes \mathsf{e}_{kl}$ we have

$$[\![M]\!]_{abcd} = \sum_{i,j,k,l=1}^{n} [M]_{i+n(a-1),j+n(b-1),k+n(c-1),l+n(d-1)}\, e_{ij} \otimes e_{kl}. \tag{3.2.32}$$

Denote the $\mathfrak{g}_{2n}$ R-matrix acting on $\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$ by $\widetilde{R}(u)$. Viewing $\widetilde{R}(u)$ as element in $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)[[u^{-1}]]$ and using (3.2.32) we recover the familiar six-vertex block structure,

$$\widetilde{R}(u) = \begin{pmatrix} R(u) & & & \\ & R^t(\kappa-u) & U(u) & \\ & U(u) & R^t(\kappa-u) & \\ & & & R(u) \end{pmatrix}. \tag{3.2.33}$$

The operators inside the matrix above are each acting on $\mathbb{C}^n \otimes \mathbb{C}^n$ and are given by

$$R(u) = I - \frac{1}{u} P, \qquad U(u) = -\frac{1}{u} P \pm \frac{1}{u-\kappa} Q, \tag{3.2.34}$$

where both the transpose t and the projector $Q = \sum_{i,j=1}^{N} e_{ij} \otimes e_{\bar{\jmath}\bar{\imath}}$ are of the orthogonal type (recall the notation $\bar{\imath} = n - i + 1$), and I is the identity matrix. These operators satisfy the following unitarity relations

$$R(u)\, R(-u) = (1 - u^{-2})\, I, \qquad R^t(u)\, R^t(n-u) = I. \tag{3.2.35}$$

In a similar way, the matrices $T_1(u)$ and $T_2(u)$, as elements of $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n) \otimes X(\mathfrak{g}_{2n})[[u^{-1}]]$, take the form

$$T_1(u) = \begin{pmatrix} \overline{A}_1(u) & & \overline{B}_1(u) & \\ & \overline{A}_1(u) & & \overline{B}_1(u) \\ \overline{C}_1(u) & & \overline{D}_1(u) & \\ & \overline{C}_1(u) & & \overline{D}_1(u) \end{pmatrix}, \quad T_2(u) = \begin{pmatrix} \overline{A}_2(u) & \overline{B}_2(u) & & \\ \overline{C}_2(u) & \overline{D}_2(u) & & \\ & & \overline{A}_2(u) & \overline{B}_2(u) \\ & & \overline{C}_2(u) & \overline{D}_2(u) \end{pmatrix}. \tag{3.2.36}$$

where $\overline{A}_1(u)$ means $\overline{A}(u) \otimes I \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n) \otimes X(\mathfrak{g}_{2n})[[u^{-1}]]$, and similarly for the other blocks. Substituting (3.2.33) and (3.2.36) to (1.2.27) allows us to rewrite the defining relations of $X(\mathfrak{g}_{2n})$

in terms of the matrices $\overline{A}(u)$, $\overline{B}(u)$, $\overline{C}(u)$ and $\overline{D}(u)$. The relations that we will need are:

$$R(u-v)\,\overline{A}_1(u)\,\overline{A}_2(v) = \overline{A}_2(v)\,\overline{A}_1(u)\,R(u-v), \tag{3.2.37}$$

$$R(u-v)\,\overline{D}_1(u)\,\overline{D}_2(v) = \overline{D}_2(v)\,\overline{D}_1(u)\,R(u-v), \tag{3.2.38}$$

$$R^t(\kappa-u+v)\,\overline{C}_1(u)\,\overline{A}_2(v) = \overline{A}_2(v)\,\overline{C}_1(u)\,R(u-v) + Q(u-v)\,\overline{A}_1(u)\,\overline{C}_2(v), \tag{3.2.39}$$

$$\overline{C}_2(v)\,\overline{D}_1(u)\,R^t(\kappa-u+v) = R(u-v)\,\overline{D}_1(u)\,\overline{C}_2(v) - \overline{D}_2(v)\,\overline{C}_1(u)\,K(u-v), \tag{3.2.40}$$

$$\begin{aligned}\overline{A}_2(v)\,\overline{D}_1(u)\,R^t(\kappa-u+v) &- R^t(\kappa-u+v)\,\overline{D}_1(u)\,\overline{A}_2(v)\\ &= Q(u-v)\,\overline{B}_1(u)\,\overline{C}_2(v) - \overline{B}_2(v)\,\overline{C}_1(u)\,Q(u-v).\end{aligned} \tag{3.2.41}$$

In particular, the coefficients of the matrix entries of $\overline{A}(u)$ generate a $Y(\mathfrak{gl}_n)$ subalgebra of $X(\mathfrak{g}_{2n})$. The same is true for $\overline{D}(u)$.

We now repeat the same steps for the extended twisted Yangian $X_\rho(\mathfrak{g}_{2n},\mathfrak{g}_{2n}^\theta)^{tw}$. We substitute (3.2.33) to (3.2.10) and view the matrices $S_1(u)$ and $S_2(u)$ as elements of $\mathrm{End}(\mathbb{C}^2\otimes\mathbb{C}^2)\otimes\mathrm{End}(\mathbb{C}^n\otimes\mathbb{C}^n)\otimes X_\rho(\mathfrak{g}_{2n},\mathfrak{g}_{2n}^\theta)^{tw}((u^{-1}))$, so that they take the same form as in (3.2.36). This allows us to write the defining relations of $X_\rho(\mathfrak{g}_{2n},\mathfrak{g}_{2n}^\theta)^{tw}$ in terms of the matrices $A(u)$, $B(u)$, $C(u)$ and $D(u)$. The relations that we will need are:

$$\begin{aligned}R(u-v)\,A_1(u)\,R(u+v+\rho)\,A_2(v) &= A_2(v)\,R(u+v+\rho)\,A_1(u)\,R(u-v)\\ &- R(u-v)\,B_1(u)\,Q(u+v+\rho)\,C_2(v) + B_2(v)\,Q(u+v+\rho)\,C_1(u)\,R(u-v),\end{aligned} \tag{3.2.42}$$

$$\begin{aligned}A_2(v)\,R(u+v+\rho)\,B_1(u)\,Q(u-v) &= R(u-v)\,B_1(u)\,Q(u+v+\rho)\,A_2(v)\\ &- B_2(v)\,Q(u+v+\rho)\,A_1(u)\,Q(u-v) - B_2(v)\,Q(u+v+\rho)\,D_1(u)\,Q(u-v),\end{aligned} \tag{3.2.43}$$

$$\begin{aligned}R^t(\tilde{u}-v)\,C_1(u)\,R(u+v+\rho)\,A_1(v) &= A_1(v)\,R^t(\tilde{u}-v)\,C_1(u)\,R(u-v)\\ &- Q(u-v)\,A_1(u)\,R^t(\tilde{u}-v)\,C_2(v) - R^t(\tilde{u}-v)D_1(u)\,Q(u+v+\rho)\,C_1(v),\end{aligned} \tag{3.2.44}$$

$$R(u-v)\,B_1(u)\,R^t(\tilde{u}-v)\,B_2(v) = B_2(v)\,R^t(\tilde{u}-v)\,B_1(u)\,R(u-v). \tag{3.2.45}$$

It remains to cast the symmetry relation (3.2.14) in the block form. Observe that

$$S^t(u) = \begin{pmatrix} D^t(u) & \pm B^t(u) \\ \pm C^t(u) & A^t(u) \end{pmatrix}, \tag{3.2.46}$$

which allows us to immediately extract linear relations between the operators $A(u), B(u), C(u)$ and $D(u)$, of which we will need the following two only:

$$D^t(u) = -\left(1 \pm \frac{1}{u-\tilde{u}}\right)A(\tilde{u}) \pm \frac{A(u)}{u-\tilde{u}} - \frac{\mathrm{tr}(S(u))\cdot I}{u-\tilde{u}-\kappa}, \tag{3.2.47}$$

$$B^t(u) = \left(\mp 1 - \frac{1}{u-\tilde{u}}\right)B(\tilde{u}) + \frac{B(u)}{u-\tilde{u}}. \tag{3.2.48}$$

3.2.4 Creation operator for a single excitation

We begin by reinterpreting the B operator of the generating matrix $S(u)$, viz. (3.2.30), as a row vector in two auxiliary spaces, $V^*_{\tilde{a}_1} \otimes V^*_{a_1} \cong (\mathbb{C}^n)^* \otimes (\mathbb{C}^n)^*$, with components given by the matrix elements $\mathscr{b}_{ij}(u)$.

Definition 3.2.9. *The creation operator for a single (top-level) excitation is given by*

$$\beta_{\tilde{a}_1 a_1}(u) := \sum_{i,j=1}^{N} e_i^* \otimes e_j^* \otimes \mathscr{b}_{\bar{\imath}j}(u) \in V^*_{\tilde{a}_1} \otimes V^*_{a_1} \otimes X_\rho(\mathfrak{g}_{2n}, \mathfrak{g}_{2n}^\theta)^{tw}((u^{-1})). \tag{3.2.49}$$

The exchange and symmetry relations involving the B operator must now be rewritten using the above notation. In general, we may switch between the two notations using the following relation, in matrix elements,

$$(X_a B_a(u) Y_a)_{ij} = \sum_{1\le k,l\le n} x_{ik}\, \mathscr{b}_{kl}(u)\, y_{lj} = (\beta_{\tilde{a}a}(u) X^t_{\tilde{a}} Y_a)_{\bar{\imath}j}, \tag{3.2.50}$$

where X, Y are matrix operators with entries in $\mathbb{C}((u^{-1}))$ and may act nontrivially on the additional auxiliary spaces. The Lemma below states some useful properties of the creation operator.

Lemma 3.2.10. *The (top-level) creation operator satisfies the following two identities:*

$$\begin{aligned}
&\beta_{\tilde{a}_1 a_1}(u_1)\,\beta_{\tilde{a}_2 a_2}(u_2)\, R_{a_1\tilde{a}_2}(-u_1-u_2-\rho)\, \check{R}_{\tilde{a}_1\tilde{a}_2}(u_1-u_2) \\
&\qquad\qquad = \beta_{\tilde{a}_1 a_1}(u_2)\,\beta_{\tilde{a}_2 a_2}(u_1)\, R_{a_1\tilde{a}_2}(-u_1-u_2-\rho)\, \check{R}_{a_1 a_2}(u_1-u_2),
\end{aligned} \tag{3.2.51}$$

$$\beta_{\tilde{a}_1 a_1}(v)\, Q_{\tilde{a}_1 a}\, Q_{a_1 a} = \left(\mp 1 - \frac{1}{v-\tilde{v}}\right) \beta_{\tilde{a}_1 a_1}(\tilde{v})\, Q_{a_1 a} + \frac{\beta_{\tilde{a}_1 a_1}(v)\, Q_{a_1 a}}{v-\tilde{v}}. \tag{3.2.52}$$

Proof. The operator $B(u)$ satisfies the same exchange relation as the equivalent operator from Chapter 2, with an additional shift of κ. Following Lemma 2.2.4, with ρ replaced by $\rho-\kappa$, we arrive at (3.2.51). To prove (3.2.52) we work from (3.2.48). Acting from the right by $Q_{a_1 a}$, and using the equalities $X^t_{a_1} Q_{a_1 a} = X_a Q_{a_1 a} = P_{a_1 a} X_{a_1} Q_{a_1 a}$, we obtain

$$P_{a_1 a} B_{a_1}(v)\, Q_{a_1 a} = \left(\mp 1 - \frac{1}{v-\tilde{v}}\right) B_{a_1}(\tilde{v})\, Q_{a_1 a} + \frac{B_{a_1}(v)\, Q_{a_1 a}}{v-\tilde{v}}.$$

Implementing (3.2.50) then yields the desired result. □

3.2.5 The AB exchange relation for a single excitation

Our next step is to rewrite the AB exchange relation (3.2.43) in terms of the creation operator (3.2.49). Typically, the Bethe ansatz method would also require us to consider the DB exchange relation but, just as in Chapter 2, we will make use of the linear symmetry relation (3.2.47) to replace all instances of D operators by A operators. Indeed, the Lemma below allows us to rewrite the trace of the monodromy matrix $\operatorname{tr} S(u) = \operatorname{tr} A(u) + \operatorname{tr} D(u)$ in terms of the A operator only.

We will often make use of the following notation. For a function f we define a symmetrization operation by

$$\{f(u)\}^u := f(u) + f(\tilde{u}), \tag{3.2.53}$$

noting that now $\tilde{u} = \kappa - u - \rho$. We also introduce the rational function

$$p(u) = \frac{1}{u - \tilde{u}}. \tag{3.2.54}$$

Lemma 3.2.11. *We have*

$$-\frac{\operatorname{tr} S(\tilde{u})}{u - \tilde{u} + \kappa} = \frac{\operatorname{tr} S(u)}{u - \tilde{u} - \kappa} = \{p(u) \operatorname{tr} A(u)\}^u .$$

Proof. Adding $A(u)$ to both sides of (3.2.47) and taking the trace we obtain

$$\operatorname{tr} S(u) = \left(1 \pm \frac{1}{u - \tilde{u}}\right) \operatorname{tr}(A(u) - A(\tilde{u})) - \frac{(\kappa \pm 1) \operatorname{tr} S(u)}{u - \tilde{u} - \kappa}.$$

Rearranging this, and dividing by $(u - \tilde{u} \pm 1)$, we find

$$\frac{\operatorname{tr} S(u)}{u - \tilde{u} - \kappa} = \frac{\operatorname{tr}(A(u) - A(\tilde{u}))}{u - \tilde{u}},$$

which, by (3.2.53), proves the second equality. The first equality is obtained by sending $u \mapsto \tilde{u}$, and noting that the r.h.s. remains unchanged. □

Applying Lemma 3.2.11 to (3.2.47) we obtain a new symmetry relation for the A and D operators:

$$D^t(u) = -\left(1 \pm \frac{1}{u - \tilde{u}}\right) A(\tilde{u}) \pm \frac{A(u)}{u - \tilde{u}} - \left\{\frac{\operatorname{tr}(A(u)) \cdot I}{u - \tilde{u}}\right\}^u. \tag{3.2.55}$$

This symmetry relation allows us to obtain a D-independent form of the AB exchange relation.

Lemma 3.2.12. *The AB exchange relation* (3.2.43) *may be equivalently written as*

$$A_a(v)\, \beta_{\tilde{a}_1 a_1}(u) = \beta_{\tilde{a}_1 a_1}(u)\, S^{(1)}_{a \tilde{a}_1 a_1}(v; u) + U^+ + U^-, \tag{3.2.56}$$

where

$$S^{(1)}_{a \tilde{a}_1 a_1}(v; u) := R^t_{\tilde{a}_1 a}(u - v)\, R^t_{a_1 a}(\tilde{u} - v)\, A_a(v)\, R^t_{a_1 a}(u - v \pm 1)\, R^t_{\tilde{a}_1 a}(\tilde{u} - v \pm 1), \tag{3.2.57}$$

$$U^+ := \frac{\beta_{\tilde{a}_1 a_1}(v)}{u - v} Q_{\tilde{a}_1 a}\, R^t_{a_1 a}(\tilde{u} - u)\, A_a(u)\, R^t_{a_1 a}(\pm 1)\, R^t_{\tilde{a}_1 a}(\tilde{u} - u \pm 1), \tag{3.2.58}$$

$$U^- := \pm \frac{\beta_{\tilde{a}_1 a_1}(v)}{u - \tilde{v}} Q_{\tilde{a}_1 a} Q_{a_1 a} \left(1 \pm \frac{1}{u - \tilde{u}}\right) A_a(\tilde{u})\, R^t_{a_1 a}(u - \tilde{u} \pm 1)\, R^t_{\tilde{a}_1 a}(\pm 1). \tag{3.2.59}$$

The matrix $S^{(1)}_{a \tilde{a}_1 a_1}(v; u)$ is the nested monodromy matrix for a single excitation. The matrices

$U^{\pm}$ are the 'unwanted terms', which will also be written as "UWT".

Proof of Lemma 3.2.12. The first step is to rewrite (3.2.43) in terms of $\beta_{\tilde{a}_1 a_1}(u)$. We obtain, using (3.2.50),

$$
\begin{aligned}
&A_a(v)\,\beta_{\tilde{a}_1 a_1}(u)\,R^t_{\tilde{a}_1 a}(u+v+\rho)\,R^t_{a_1 a}(\kappa-u+v)\\
&\quad=\beta_{\tilde{a}_1 a_1}(u)\,R^t_{\tilde{a}_1 a}(u-v)\,R^t_{a_1 a}(\tilde{u}-v)\,A_a(v)\\
&\qquad-\beta_{\tilde{a}_1 a_1}(v)Q_{\tilde{a}_1 a}\,P_{a_1 a}\,R^t_{a_1 a}(\tilde{u}-v)\,A_{a_1}(u)\,U_{a_1 a}(u-v)\\
&\qquad-\beta_{\tilde{a}_1 a_1}(v)Q_{\tilde{a}_1 a}\,P_{a_1 a}U_{a_1 a}(u+v+\rho)\,D_{a_1}(u)\,R^t_{a_1 a}(\kappa-u+v).
\end{aligned}
$$

Since Q is a rank n projector, $R^t(u)$ is invertible for $u \neq n$, with inverse $R^t(n-u) = R^t(\kappa - u \pm 1)$. Multiplying the expression above by the appropriate inverses, we have

$$
\begin{aligned}
&A_a(v)\,\beta_{\tilde{a}_1 a_1}(u)\\
&\quad=\beta_{\tilde{a}_1 a_1}(u)\,R^t_{\tilde{a}_1 a}(u-v)\,R^t_{a_1 a}(\tilde{u}-v)\,A_a(v)\,R^t_{a_1 a}(u-v\pm 1)\,R^t_{\tilde{a}_1 a}(\tilde{u}-v\pm 1)\\
&\qquad-\beta_{\tilde{a}_1 a_1}(v)Q_{\tilde{a}_1 a}\,P_{a_1 a}\,R^t_{a_1 a}(\tilde{u}-v)\,A_{a_1}(u)\,U_{a_1 a}(u-v)R^t_{a_1 a}(u-v\pm 1)\,R^t_{\tilde{a}_1 a}(\tilde{u}-v\pm 1)\\
&\qquad-\beta_{\tilde{a}_1 a_1}(v)Q_{\tilde{a}_1 a}\,P_{a_1 a}U_{a_1 a}(u+v+\rho)\,D_{a_1}(u)\,R^t_{\tilde{a}_1 a}(\tilde{u}-v\pm 1)\\
&\quad=\beta_{\tilde{a}_1 a_1}(u)\,S^{(1)}_{a\,\tilde{a}_1 a_1}(v;u)+U^A+U^D,
\end{aligned}
\tag{3.2.60}
$$

where

$$
\begin{aligned}
U^A &:= -\beta_{\tilde{a}_1 a_1}(v)\,Q_{\tilde{a}_1 a}\,P_{a_1 a}\,R^t_{a_1 a}(\tilde{u}-v)\,A_{a_1}(u)U_{a_1 a}(u-v)R^t_{a_1 a}(u-v\pm 1)\,R^t_{\tilde{a}_1 a}(\tilde{u}-v\pm 1),\\
U^D &:= -\beta_{\tilde{a}_1 a_1}(v)\,Q_{\tilde{a}_1 a}\,P_{a_1 a}U_{a_1 a}(u+v+\rho)\,D_{a_1}(u)\,R^t_{\tilde{a}_1 a}(\tilde{u}-v\pm 1).
\end{aligned}
$$

So far the first term, the "wanted term", matches the desired expression (3.2.56). We must now manipulate $U^A + U^D$ to match the remaining terms. First, note that

$$
\begin{aligned}
U(u-v)\,R^t(u-v\pm 1) &= \left(-\frac{P}{u-v} \pm \frac{Q}{u-v-\kappa}\right)\left(1-\frac{Q}{u-v\pm 1}\right)\\
&= -\frac{P}{u-v} + \left(\frac{1}{(u-v)(u-v\pm 1)} \pm \frac{1}{u-v-\kappa}\left(1-\frac{\kappa\pm 1}{u-v\pm 1}\right)\right)Q\\
&= -\frac{P}{u-v} + \left(\frac{1}{(u-v)(u-v\pm 1)} \pm \frac{1}{u-v\pm 1}\right)Q\\
&= -\frac{P\,R^t(\pm 1)}{u-v}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
U^A &= \frac{\beta_{\tilde{a}_1 a_1}(v)}{u-v}\,Q_{\tilde{a}_1 a}\,P_{a_1 a}\,R^t_{a_1 a}(\tilde{u}-v)\,A_{a_1}(u)\,P_{a_1 a}\,R^t_{a_1 a}(\pm 1)\,R^t_{\tilde{a}_1 a}(\tilde{u}-v\pm 1)\\
&= \frac{\beta_{\tilde{a}_1 a_1}(v)}{u-v}\,Q_{\tilde{a}_1 a}\,R^t_{a_1 a}(\tilde{u}-v)\,A_a(u)\,R^t_{a_1 a}(\pm 1)\,R^t_{\tilde{a}_1 a}(\tilde{u}-v\pm 1).
\end{aligned}
\tag{3.2.61}
$$

With U^D, our strategy will be to use the symmetry relation (3.2.55), allowing us to combine the term with U^A. We first make the following manipulations in preparation:

$$\begin{aligned}
U^D &= -\beta_{\tilde{a}_1 a_1}(v) Q_{\tilde{a}_1 a} P_{a_1 a} U_{a_1 a}(u+v+\rho) R^t_{\tilde{a}_1 a}(\tilde{u}-v\pm 1) D_{a_1}(u) \\
&= -\beta_{\tilde{a}_1 a_1}(v) Q_{\tilde{a}_1 a}\left(-\frac{1}{u+v+\rho} \pm \frac{Q_{a_1 a}}{u-\tilde{v}}\right)\left(1+\frac{Q_{\tilde{a}_1 a}}{u-\tilde{v}\mp 1}\right) D_{a_1}(u) \\
&= -\beta_{\tilde{a}_1 a_1}(v)\left(-\frac{Q_{\tilde{a}_1 a}}{u+v+\rho} \pm \frac{Q_{\tilde{a}_1 a} Q_{a_1 a}}{u-\tilde{v}} + \frac{Q_{\tilde{a}_1 a}}{u-\tilde{v}\mp 1}\left(-\frac{\kappa\pm 1}{u+v+\rho} \pm \frac{1}{u-\tilde{v}}\right)\right) D_{a_1}(u).
\end{aligned}$$

The final term may be factorised as follows:

$$-\frac{\kappa\pm 1}{u+v+\rho} \pm \frac{1}{u-\tilde{v}} = -\frac{\kappa(u-\tilde{v}\mp 1)}{(u+v+\rho)(u-\tilde{v})}.$$

Thus,

$$\begin{aligned}
U^D &= -\beta_{\tilde{a}_1 a_1}(v)\left(-\frac{Q_{\tilde{a}_1 a}}{u+v+\rho} \pm \frac{Q_{\tilde{a}_1 a} Q_{a_1 a}}{u-\tilde{v}} - \frac{\kappa Q_{\tilde{a}_1 a}}{(u+v+\rho)(u-\tilde{v})}\right) D_{a_1}(u) \\
&= -\beta_{\tilde{a}_1 a_1}(v)\left(\pm\frac{Q_{\tilde{a}_1 a} Q_{a_1 a}}{u-\tilde{v}} - \frac{Q_{\tilde{a}_1 a}}{u-\tilde{v}}\right) D_{a_1}(u) \\
&= -\frac{\beta_{\tilde{a}_1 a_1}(v)}{u-\tilde{v}}\big(\pm Q_{\tilde{a}_1 a} Q_{a_1 a} - Q_{\tilde{a}_1 a}\big) D_{a_1}(u).
\end{aligned}$$

Although this expression is now a lot simpler, the $D_{a_1}(u)$ operator is acting on the auxiliary space V_{a_1}, rather than V_a as desired. To remedy this, we use the following identity:

$$Q_{\tilde{a}_1 a} D_{a_1}(u) = Q_{\tilde{a}_1 a}\,\mathrm{tr}_a(Q_{a_1 a} D_{a_1}(u)) = Q_{\tilde{a}_1 a}\,\mathrm{tr}_a(Q_{a_1 a} D^t_a(u)) = Q_{\tilde{a}_1 a} Q_{a_1 a} D^t_a(u) Q_{\tilde{a}_1 a},$$

where we have used that $\mathrm{tr}_a(Q_{a_1 a}) = I_{a_1}$. Therefore, using also $Q_{a_1 a} D_{a_1}(u) = Q_{a_1 a} D^t_a(u)$,

$$U^D = \mp\frac{\beta_{\tilde{a}_1 a_1}(v)}{u-\tilde{v}} Q_{\tilde{a}_1 a} Q_{a_1 a} D^t_a(u)(1\mp Q_{\tilde{a}_1 a}).$$

Applying the symmetry relation (3.2.55) we obtain

$$U^D = \mp\frac{\beta_{\tilde{a}_1 a_1}(v)}{u-\tilde{v}} Q_{\tilde{a}_1 a} Q_{a_1 a}\left(-\left(1\pm\frac{1}{u-\tilde{u}}\right) A_a(\tilde{u}) \pm \frac{A_a(u)}{u-\tilde{u}} - \left\{\frac{\mathrm{tr}(A(u))\cdot I_a}{u-\tilde{u}}\right\}^u\right)(1\mp Q_{\tilde{a}_1 a}).$$

Since all terms in $U^A + U^D$ contain $A_a(u)$ or $A_a(\tilde{u})$, we reorganise the sum $U^A + U^D$ accordingly.

Define

$$
\begin{aligned}
U^+ &:= \frac{\beta_{\tilde{a}_1 a_1}(v)}{u-v} Q_{\tilde{a}_1 a}\, R^t_{a_1 a}(\tilde{u}-v)\, A_a(u)\, R^t_{a_1 a}(\pm 1)\, R^t_{\tilde{a}_1 a}(\tilde{u}-v\pm 1) \\
&\qquad\qquad - \frac{\beta_{\tilde{a}_1 a_1}(v)}{u-\tilde{v}} Q_{\tilde{a}_1 a} Q_{a_1 a} \left(\frac{A_a(u) \mp \mathrm{tr}(A(u))}{u-\tilde{u}} \right)(1 \mp Q_{\tilde{a}_1 a}), \\
U^- &:= \pm \frac{\beta_{\tilde{a}_1 a_1}(v)}{u-\tilde{v}} Q_{\tilde{a}_1 a} Q_{a_1 a} \left(\left(1 \pm \frac{1}{u-\tilde{u}}\right) A_a(\tilde{u}) - \frac{\mathrm{tr}(A(\tilde{u}))}{u-\tilde{u}} \right)(1 \mp Q_{\tilde{a}_1 a}),
\end{aligned}
$$

so that $U^A + U^D = U^+ + U^-$. It remains to match the expressions for U^+ and U^- with those in the desired expressions (3.2.58), (3.2.59). With U^-, we simply use $Q_{a_1 a}\, \mathrm{tr}(A(\tilde{u})) = Q_{a_1 a}\, A_a(\tilde{u})\, Q_{a_1 a}$ to obtain the required form,

$$
U^- = \pm \frac{\beta_{\tilde{a}_1 a_1}(v)}{u-\tilde{v}} Q_{\tilde{a}_1 a} Q_{a_1 a} \left(1 \pm \frac{1}{u-\tilde{u}}\right) A_a(\tilde{u})\, R^t_{a_1 a}(u-\tilde{u}\pm 1)\, R^t_{\tilde{a}_1 a}(\pm 1).
$$

We now turn our attention to U^+. Using again the trace property of $Q_{a_1 a}$,

$$
\begin{aligned}
U^+ &= \frac{\beta_{\tilde{a}_1 a_1}(v)}{u-v} Q_{\tilde{a}_1 a}\, R^t_{a_1 a}(\tilde{u}-v)\, A_a(u)\, R^t_{a_1 a}(\pm 1)\, R^t_{\tilde{a}_1 a}(\tilde{u}-v\pm 1) \\
&\qquad\qquad - \frac{\beta_{\tilde{a}_1 a_1}(v)}{u-\tilde{v}} Q_{\tilde{a}_1 a} Q_{a_1 a} \frac{A_a(u)}{u-\tilde{u}} R^t_{a_1 a}(\pm 1)(1 \mp Q_{\tilde{a}_1 a}).
\end{aligned}
$$

The $Q_{\tilde{a}_1 a}$ and $R^t_{a_1 a}(\pm 1)$ matrices are present in both terms as desired. The simplest way forward is to expand the remaining matrices in terms of projectors, then match term by term. Indeed,

$$
\begin{aligned}
U^+ &= \beta_{\tilde{a}_1 a_1}(v)\, Q_{\tilde{a}_1 a} \Bigg(\frac{1}{u-v} A_a(u) R^t_{a_1 a}(\pm 1) + \frac{A_a(u)\, R^t_{a_1 a}(\pm 1)\, Q_{\tilde{a}_1 a}}{(u-v)(u-\tilde{v}\mp 1)} \\
&\qquad + \frac{1}{u-\tilde{v}} \left(\frac{1}{u-v} - \frac{1}{u-\tilde{u}} \right) Q_{a_1 a} A_a(u) R^t_{a_1 a}(\pm 1) \\
&\qquad \pm \frac{1}{u-\tilde{v}} \left(\frac{1}{u-\tilde{u}} \pm \frac{1}{(u-v)(u-\tilde{v}\mp 1)} \right) Q_{a_1 a} A_a(u) R^t_{a_1 a}(\pm 1) Q_{\tilde{a}_1 a} \Bigg) \\
&= \beta_{\tilde{a}_1 a_1}(v)\, Q_{\tilde{a}_1 a} \Bigg(\frac{1}{u-v} A_a(u) R^t_{a_1 a}(\pm 1) + \frac{A_a(u)\, R^t_{a_1 a}(\pm 1)\, Q_{\tilde{a}_1 a}}{(u-v)(u-\tilde{v}\mp 1)} \\
&\qquad \pm \frac{u-v\pm 1}{(u-\tilde{u})(u-v)(u-\tilde{v}\mp 1)} Q_{a_1 a} A_a(u) R^t_{a_1 a}(\pm 1) Q_{\tilde{a}_1 a} \Bigg).
\end{aligned}
$$

Although all the terms have been fully written out, it is still not clear that this is equal to the desired expression. The discrepancy arises due to the terms on the last line. These terms contain two $Q_{\tilde{a}_1 a}$ operators, and so elements sandwiched between these operators appear as a trace. This leads to the following identities

$$
\begin{aligned}
Q_{\tilde{a}_1 a} Q_{a_1 a} A_a(u) Q_{a_1 a} Q_{\tilde{a}_1 a} &= Q_{\tilde{a}_1 a} A_a(u) Q_{\tilde{a}_1 a}, \\
Q_{\tilde{a}_1 a} Q_{a_1 a} A_a(u) Q_{\tilde{a}_1 a} &= Q_{\tilde{a}_1 a} A_a(u) Q_{a_1 a} Q_{\tilde{a}_1 a}.
\end{aligned} \tag{3.2.62}
$$

Then, expanding the $R^t_{a_1a}(\pm 1)$ matrices,

$$\begin{aligned}U^+ &= \frac{\beta_{\tilde{a}_1a_1}(v)}{u-v}Q_{\tilde{a}_1a}R^t_{a_1a}(\tilde{u}-u)A_a(u)R^t_{a_1a}(\pm 1)\\ &\quad + \beta_{\tilde{a}_1a_1}(v)\big(\alpha_1 Q_{\tilde{a}_1a}Q_{a_1a}A_a(u)Q_{\tilde{a}_1a} + \alpha_2 Q_{\tilde{a}_1a}Q_{a_1a}A_a(u)Q_{a_1a}Q_{\tilde{a}_1a}\big),\end{aligned}$$

where we find

$$\alpha_1 = \mp\alpha_2 = \mp\frac{1}{(u-v)(u-\tilde{v}\mp 1)} \pm \frac{u-v\pm 1}{(u-\tilde{u})(u-v)(u-\tilde{v}\mp 1)} = \mp\frac{1}{(u-v)(u-\tilde{u})}.$$

So

$$\begin{aligned}U^+ = \frac{\beta_{\tilde{a}_1a_1}(v)}{u-v}\Bigg(&Q_{\tilde{a}_1a}R^t_{a_1a}(\tilde{u}-u)A_a(u)R^t_{a_1a}(\pm 1)\\ &\mp\frac{Q_{\tilde{a}_1a}Q_{a_1a}A_a(u)Q_{\tilde{a}_1a}}{u-\tilde{u}} + \frac{Q_{\tilde{a}_1a}Q_{a_1a}A_a(u)Q_{a_1a}Q_{\tilde{a}_1a}}{u-\tilde{u}}\Bigg).\end{aligned}$$

Writing

$$\frac{1}{u-\tilde{u}} = \frac{1}{u-\tilde{u}}\cdot\frac{u-\tilde{u}\mp 1}{u-\tilde{u}\mp 1} = \frac{1}{u-\tilde{u}\mp 1}\left(1\mp\frac{1}{u-\tilde{u}}\right),$$

and using (3.2.62) we obtain

$$\begin{aligned}U^+ &= \frac{\beta_{\tilde{a}_1a_1}(v)}{u-v}\Bigg(Q_{\tilde{a}_1a}R^t_{a_1a}(\tilde{u}-u)A_a(u)R^t_{a_1a}(\pm 1)\mp\frac{Q_{\tilde{a}_1a}A_a(u)Q_{a_1a}Q_{\tilde{a}_1a}}{u-\tilde{u}\mp 1}\\ &\qquad + \frac{Q_{\tilde{a}_1a}Q_{a_1a}A_a(u)Q_{\tilde{a}_1a}}{(u-\tilde{u})(u-\tilde{u}\mp 1)} + \frac{Q_{\tilde{a}_1a}A_a(u)Q_{\tilde{a}_1a}}{u-\tilde{u}\mp 1}\mp\frac{Q_{\tilde{a}_1a}Q_{a_1a}A_a(u)Q_{a_1a}Q_{\tilde{a}_1a}}{(u-\tilde{u})(u-\tilde{u}\mp 1)}\Bigg)\\ &= \frac{\beta_{\tilde{a}_1a_1}(v)}{u-v}Q_{\tilde{a}_1a}\,R^t_{a_1a}(\tilde{u}-u)\,A_a(u)\,R^t_{a_1a}(\pm 1)\,R^t_{\tilde{a}_1a}(\tilde{u}-u\pm 1),\end{aligned}$$

which matches (3.2.58), as required. □

From Lemma 3.2.12, in order to obtain the most elegant form of the unwanted terms, we must symmetrise over $v\to\tilde{v}$. This will allow us to write the unwanted terms as a residue of the wanted terms. We will employ the notation (3.2.53).

Lemma 3.2.13. *The AB exchange relation for a single excitation is*

$$\begin{aligned}\big\{p(v)\,A_a(v)\big\}^v\beta_{\tilde{a}_1a_1}(u) &= \beta_{\tilde{a}_1a_1}(u)\big\{p(v)\,S^{(1)}_{a\,\tilde{a}_1a_1}(v;u)\big\}^v\\ &\quad + \frac{1}{p(u)}\left\{p(v)\,\frac{\beta_{\tilde{a}_1a_1}(v)}{u-v}\right\}^v\operatorname*{Res}_{w\to u}\big\{p(w)\,S^{(1)}_{a\,\tilde{a}_1a_1}(w;u)\big\}^w.\end{aligned}$$

Proof. Evaluating the residue, the desired expression is

$$\begin{aligned}
&\{p(v)A_a(v)\}^v\beta_{\tilde{a}_1a_1}(u)\\
&\quad=\beta_{\tilde{a}_1a_1}(u)\{p(v)\,R^t_{\tilde{a}_1a}(u-v)\,R^t_{a_1a}(\tilde{u}-v)\,A_a(v)\,R^t_{a_1a}(u-v\pm1)\,R^t_{\tilde{a}_1a}(\tilde{u}-v\pm1)\}^v\\
&\qquad+\left\{p(v)\frac{\beta_{\tilde{a}_1a_1}(v)}{u-v}\right\}^v Q_{\tilde{a}_1a}\,R^t_{a_1a}(\tilde{u}-u)\,A_a(u)\,R^t_{a_1a}(\pm1)\,R^t_{\tilde{a}_1a}(\tilde{u}-u\pm1)\\
&\qquad+\left\{p(v)\frac{\beta_{\tilde{a}_1a_1}(v)}{u-v}\right\}^v R^t_{\tilde{a}_1a}(u-\tilde{u})\,Q_{a_1a}\,A_a(\tilde{u})\,R^t_{a_1a}(u-\tilde{u}\pm1)\,R^t_{\tilde{a}_1a}(\pm1).
\end{aligned}\tag{3.2.63}$$

We will work from (3.2.56), and obtain this expression. Multiplying (3.2.56) by $p(v)$ and symmetrising over $v\to\tilde{v}$ reveals that the "wanted term" and U^+ term are already of the correct form, while U^- is of the form

$$\{p(v)U^-\}^v=\pm\left\{p(v)\frac{\beta_{\tilde{a}_1a_1}(v)}{u-\tilde{v}}\right\}^v Q_{\tilde{a}_1a}Q_{a_1a}\left(1\pm\frac{1}{u-\tilde{u}}\right)A_a(\tilde{u})\,R^t_{a_1a}(u-\tilde{u}\pm1)\,R^t_{\tilde{a}_1a}(\pm1).$$

From here, we will use the identity (3.2.52) to construct $R^t(u-\tilde{u})$, and arrive at the desired expression. We must split the r.h.s. into two portions, on one of which we will use the to construct the "identity" part of the R^t-matrix. Combining this with the other portion will result in the desired R^t-matrix. It turns out the correct proportions to take are given as follows:

$$\begin{aligned}
\frac{1}{u-\tilde{v}}\left(1\pm\frac{1}{u-\tilde{u}}\right)&=\frac{1}{u-\tilde{v}}\left(1\pm\frac{1}{u-v}\mp\frac{1}{u-v}\pm\frac{1}{u-\tilde{u}}\right)\\
&=\frac{1}{u-\tilde{v}}\left(1\pm\frac{1}{u-v}\mp\frac{u-\tilde{v}}{(u-v)(u-\tilde{u})}\right)\\
&=\frac{1}{u-\tilde{v}}\left(1\pm\frac{1}{u-v}\right)\mp\frac{1}{(u-v)(u-\tilde{u})}.
\end{aligned}$$

Then,

$$\begin{aligned}
\{p(v)U^-\}^v=\pm&\left\{p(v)\,\frac{\beta_{\tilde{a}_1a_1}(v)}{u-v}\left(\frac{u-v\pm1}{u-\tilde{v}}\mp\frac{1}{u-\tilde{u}}\right)\right\}^v\\
&\times Q_{\tilde{a}_1a}Q_{a_1a}A_a(\tilde{u})\,R^t_{a_1a}(u-\tilde{u}\pm1)\,R^t_{\tilde{a}_1a}(\pm1).
\end{aligned}$$

Applying (3.2.52) to the first of these terms, we have

$$
\begin{aligned}
\pm & \left\{ p(v)\, \frac{\beta_{\tilde{a}_1 a_1}(v)}{u-v} \left(\frac{u-v\pm 1}{u-\tilde{v}} \right) \right\}^{v} Q_{\tilde{a}_1 a} Q_{a_1 a} \\
&= \pm \left\{ \frac{p(v)}{u-v} \left(\frac{u-v\pm 1}{u-\tilde{v}} \right) \left(\left(\mp 1 - \frac{1}{v-\tilde{v}} \right) \beta_{\tilde{a}_1 a_1}(\tilde{v})\, Q_{a_1 a} + \frac{\beta_{\tilde{a}_1 a_1}(v)\, Q_{a_1 a}}{v-\tilde{v}} \right) \right\}^{v} \\
&= \pm \left\{ p(v) \frac{\beta_{\tilde{a}_1 a_1}(v)}{(u-v)(u-\tilde{v})} \left((u-\tilde{v}\pm 1) \left(\pm 1 - \frac{1}{v-\tilde{v}} \right) + \frac{u-v\pm 1}{u-\tilde{v}} \right) \right\}^{v} Q_{a_1 a} \\
&= \left\{ p(v) \frac{\beta_{\tilde{a}_1 a_1}(v)}{(u-v)(u-\tilde{v})} \left(\frac{(u-\tilde{v}\pm 1)(v-\tilde{v}\mp 1) \pm (u-v\pm 1)}{v-\tilde{v}} \right) \right\}^{v} Q_{a_1 a} \\
&= \left\{ p(v) \frac{\beta_{\tilde{a}_1 a_1}(v)}{(u-v)(u-\tilde{v})} \left(\frac{(v-\tilde{v})(u-\tilde{v})}{v-\tilde{v}} \right) \right\}^{v} Q_{a_1 a} \\
&= \left\{ p(v) \frac{\beta_{\tilde{a}_1 a_1}(v)}{u-v} \right\}^{v} Q_{a_1 a}.
\end{aligned}
$$

Therefore

$$
\left\{ p(v) U^{-} \right\}^{v} = \left\{ p(v)\, \frac{\beta_{\tilde{a}_1 a_1}(v)}{u-v} \right\}^{v} R^{t}_{\tilde{a}_1 a}(u-\tilde{u}) Q_{a_1 a}\, A_a(\tilde{u})\, R^{t}_{a_1 a}(u-\tilde{u}\pm 1)\, R^{t}_{\tilde{a}_1 a}(\pm 1), \tag{3.2.64}
$$

which agrees with the last term in (3.2.63). □

Lemmas 3.2.12 and 3.2.13 provide us with an insight into the expression for the nested monodromy matrix of the spin chain. The next step is to generalize the result of Lemma 3.2.13 for an arbitrary number of excitations.

3.2.6 Creation operator for multiple excitations

Choose $m \in \mathbb{N}$, the number of (top-level) excitations, and introduce m-tuple $\boldsymbol{u} = (u_1, u_2, \ldots, u_m)$ of formal parameters and m-tuples $\tilde{\boldsymbol{a}} = (\tilde{a}_1, \ldots, \tilde{a}_m)$ and $\boldsymbol{a} = (a_1, \ldots, a_m)$ of labels. For each label we associate an auxiliary vector space, $V_{\tilde{a}_1}, V_{a_1}, \ldots, V_{\tilde{a}_m}, V_{a_m}$, each isomorphic to $\mathbb{C}^n$. Then we define a tensor space $W_{\tilde{\boldsymbol{a}}\boldsymbol{a}}$ and its dual $W^{*}_{\tilde{\boldsymbol{a}}\boldsymbol{a}}$ by

$$
W_{\tilde{\boldsymbol{a}}\boldsymbol{a}} := V_{\tilde{a}_1} \otimes V_{a_1} \otimes \cdots \otimes V_{\tilde{a}_m} \otimes V_{a_m}, \qquad W^{*}_{\tilde{\boldsymbol{a}}\boldsymbol{a}} := V^{*}_{\tilde{a}_1} \otimes V^{*}_{a_1} \otimes \cdots V^{*}_{\tilde{a}_m} \otimes V^{*}_{a_m}. \tag{3.2.65}
$$

Definition 3.2.14. *The creation operator for m (top-level) excitations is given by*

$$
\beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u}) := \prod_{i=1}^{m} \left(\beta_{\tilde{a}_i a_i}(u_i) \prod_{j=i-1}^{1} R_{a_j \tilde{a}_i}(\tilde{u}_i - u_j) \right) \in W^{*}_{\tilde{\boldsymbol{a}}\boldsymbol{a}} \otimes X(\mathfrak{g}_{2n}, \mathfrak{g}^{\rho}_{2n})^{tw}((u_1^{-1}, \ldots, u_m^{-1})).
$$

Note that $\beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u})$ satisfies the following recursion relation:

$$
\begin{aligned}
\beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(u_1, \ldots, u_m) &= \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_{m-1} a_{m-1}}(u_1, \ldots, u_{m-1})\, \beta_{\tilde{a}_m a_m}(u_m) \\
&\qquad \times R_{a_{m-1} \tilde{a}_m}(\tilde{u}_m - u_{m-1}) \cdots R_{a_1 \tilde{a}_m}(\tilde{u}_m - u_1).
\end{aligned}
$$

Our next step is to obtain an identity relating $\beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u})$ with $\beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u}_{i\leftrightarrow i+1})$, where $\boldsymbol{u}_{i\leftrightarrow i+1}$ denotes the m-tuple obtained from $\boldsymbol{u}$ by interchanging u_i with u_{i+1} for any $1 \le i \le m-1$. For this purpose, we define

$$\check{R}(u) := \frac{u}{u-1} PR(u).$$

The normalisation here is chosen such that $\check{R}(u)\check{R}(-u) = I$.

Lemma 3.2.15. *The creation operator for m (top-level) excitations obeys the following symmetry*

$$\beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u}) = \beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u}_{i\leftrightarrow i+1})\,\check{R}_{a_i a_{i+1}}(u_i - u_{i+1})\,\check{R}_{\tilde{a}_i\tilde{a}_{i+1}}(u_{i+1} - u_i)$$

for $1 \le i \le m-1$.

Proof. The operator $B(u)$ satisfies the same defining relations as those in Chapter 2, with an additional shift of κ. Following the same argument as in Lemma 2.2.6, with $\rho - \kappa$ instead of ρ, we arrive at the same conclusion. Finally, the normalised $\check{R}$ allows us to write $\check{R}^{-1}(u) = \check{R}(-u)$. □

3.2.7 The AB exchange relation for multiple excitations

We now generalise the single excitation nested monodromy matrix $S^{(1)}_{a\tilde{a}_1a_1}(v;u_1)$ from Lemma 3.2.13 to multiple excitations.

Definition 3.2.16. *The nested monodromy matrix for k (top-level) excitations is given by*

$$S^{(1)}_{a\tilde{a}_1a_1\ldots\tilde{a}_ka_k}(v;u_1,\ldots,u_k) := \left(\prod_{i=1}^{k} R^t_{\tilde{a}_ia}(u_i - v)\right)\left(\prod_{i=1}^{k} R^t_{a_ia}(\tilde{u}_i - v)\right) \times A_a(v)\left(\prod_{i=k}^{1} R^t_{a_ia}(u_i - v \pm 1)\right)\left(\prod_{i=k}^{1} R^t_{\tilde{a}_ia}(\tilde{u}_i - v \pm 1)\right). \quad (3.2.66)$$

We will often omit the $\tilde{a}_1a_1\ldots\tilde{a}_ka_k$ from the subscript, writing simply $S^{(1)}_a(v;u_1,\ldots,u_k)$.

Lemma 3.2.17. *The following identity holds*

$$S^{(1)}_a(v;u_1,\ldots,u_{k-1})\left(\beta_{\tilde{a}_ka_k}(u_k)\prod_{j=k-1}^{1} R_{a_j\tilde{a}_k}(\tilde{u}_k - u_j)\right) = \left(\beta_{\tilde{a}_ka_k}(u_k)\prod_{j=k-1}^{1} R_{a_j\tilde{a}_k}(\tilde{u}_k - u_j)\right)S^{(1)}_a(v;u_1,\ldots,u_k) + UWT,$$

where UWT denotes the "unwanted terms" that do not contain $A_a(v)$.

Proof. Working from the definition of $S^{(1)}_a(v;u_1,\ldots,u_{k-1})$, and commuting matrices which act on different spaces, we use Lemma 3.2.12 to obtain

$$
\begin{aligned}
&S_a^{(1)}(v; u_1, \ldots, u_{k-1})\Big(\beta_{\tilde{a}_k a_k}(u_k) \prod_{j=k-1}^{1} R_{a_j \tilde{a}_k}(\tilde{u}_k - u_j)\Big) \\
&\quad = \beta_{\tilde{a}_k a_k}(u_k)\left(\prod_{i=1}^{k-1} R^t_{\tilde{a}_i a}(u_i - v)\right)\left(\prod_{i=1}^{k-1} R^t_{a_i a}(\tilde{u}_i - v)\right) \\
&\qquad \times R^t_{\tilde{a}_k a}(u_k - v)\, R^t_{a_k a}(\tilde{u}_k - v)\, A_a(v)\, R^t_{a_k a}(u_k - v \pm 1)\, R^t_{\tilde{a}_k a}(\tilde{u}_k - v \pm 1) \\
&\qquad \times \left(\prod_{i=k-1}^{1} R^t_{a_i a}(u_i - v \pm 1)\right)\left(\prod_{i=k-1}^{1} R^t_{\tilde{a}_i a}(\tilde{u}_i - v \pm 1)\right)\left(\prod_{j=k-1}^{1} R_{a_j \tilde{a}_k}(\tilde{u}_k - u_j)\right) \\
&\quad + UWT.
\end{aligned}
$$

To obtain the result, we must move the rightmost product of R-matrices to the left, using the Yang-Baxter equation. The first move is simply commuting the rightmost product of R-matrices to the left, through the product of R^t-matrices, as there is no intersection of spaces on which these products act non-trivially.

Next, we write

$$
\begin{aligned}
&R^t_{\tilde{a}_k a}(\tilde{u}_k - v \pm 1)\left(\prod_{i=k-1}^{1} R^t_{a_i a}(u_i - v \pm 1)\right)\left(\prod_{j=k-1}^{1} R_{a_j \tilde{a}_k}(\tilde{u}_k - u_j)\right) \\
&\quad = \left[\left(\prod_{i=1}^{k-1} R_{a_i a}(u_i - v \pm 1)\right) R_{\tilde{a}_k a}(\tilde{u}_k - v \pm 1)\left(\prod_{j=k-1}^{1} R_{a_j \tilde{a}_k}(\tilde{u}_k - u_j)\right)\right]^{t_a}.
\end{aligned}
$$

From here, repeated use of the Yang-Baxter equation allows us to swap the matrices on the left with those on the right. Indeed, the Yang-Baxter equation is

$$
R_{a_i a}(u_i - v \pm 1)\, R_{\tilde{a}_k a}(\tilde{u}_k - v \pm 1)\, R_{a_i \tilde{a}_k}(\tilde{u}_k - u_i) = R_{a_i \tilde{a}_k}(\tilde{u}_k - u_i)\, R_{\tilde{a}_k a}(\tilde{u}_k - v \pm 1)\, R_{a_i a}(u_i - v \pm 1).
$$

Note that after performing each swap, the R-matrix swapped to the left commutes with the remaining product of R-matrices on the left, and similarly for the R-matrix swapped to the right. Thus

$$
\begin{aligned}
&R^t_{\tilde{a}_k a}(\tilde{u}_k - v \pm 1)\left(\prod_{i=k-1}^{1} R^t_{a_i a}(u_i - v \pm 1)\right)\left(\prod_{j=k-1}^{1} R_{a_j \tilde{a}_k}(\tilde{u}_k - u_j)\right) \\
&\quad = \left[\left(\prod_{j=k-1}^{1} R_{a_j \tilde{a}_k}(\tilde{u}_k - u_j)\right) R_{\tilde{a}_k a}(\tilde{u}_k - v \pm 1)\left(\prod_{i=1}^{k-1} R_{a_i a}(u_i - v \pm 1)\right)\right]^{t_a} \\
&\quad = \left(\prod_{j=k-1}^{1} R_{a_j \tilde{a}_k}(\tilde{u}_k - u_j)\right)\left(\prod_{i=k-1}^{1} R^t_{a_i a}(u_i - v \pm 1)\right) R^t_{\tilde{a}_k a}(\tilde{u}_k - v \pm 1).
\end{aligned}
$$

So far we have

$$
\begin{aligned}
\text{l.h.s.} = \beta_{\tilde{a}_k a_k}(u_k) &\left(\prod_{i=1}^{k-1} R^t_{\tilde{a}_i a}(u_i - v)\right)\left(\prod_{i=1}^{k-1} R^t_{a_i a}(\tilde{u}_i - v)\right) \\
&\times R^t_{\tilde{a}_k a}(u_k - v)\, R^t_{a_k a}(\tilde{u}_k - v)\, A_a(v)\, R^t_{a_k a}(u_k - v \pm 1) \\
&\times \left(\prod_{j=k-1}^{1} R_{a_j \tilde{a}_k}(\tilde{u}_k - u_j)\right)\left(\prod_{i=k-1}^{1} R^t_{a_i a}(u_i - v \pm 1)\right)\left(\prod_{i=k}^{1} R^t_{\tilde{a}_i a}(\tilde{u}_i - v \pm 1)\right) \\
&+ UWT.
\end{aligned}
$$

Note that the product of R-matrices that we were moving commutes with $R^t_{a_k a}(\tilde{u}_k - v)\, A_a(v)\, R^t_{a_k a}(u_k - v \pm 1)$. Then, moving further leftwards, we must use the Yang-Baxter relation again. Specifically, we use

$$
R^t_{a_i a}(\tilde{u}_i - v)\, R^t_{\tilde{a}_k a}(u_k - v)\, R_{a_i \tilde{a}_k}(\tilde{u}_k - u_i) = R_{a_i \tilde{a}_k}(\tilde{u}_k - u_i)\, R^t_{\tilde{a}_k a}(u_k - v)\, R^t_{a_i a}(\tilde{u}_i - v),
$$

giving

$$
\begin{aligned}
&\left(\prod_{i=1}^{k-1} R^t_{a_i a}(\tilde{u}_i - v)\right) R^t_{\tilde{a}_k a}(u_k - v)\left(\prod_{j=k-1}^{1} R_{a_j \tilde{a}_k}(\tilde{u}_k - u_j)\right) \\
&\qquad = \left(\prod_{j=k-1}^{1} R_{a_j \tilde{a}_k}(\tilde{u}_k - u_j)\right) R^t_{\tilde{a}_k a}(u_k - v)\left(\prod_{i=1}^{k-1} R^t_{a_i a}(\tilde{u}_i - v)\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\text{l.h.s.} &= \beta_{\tilde{a}_k a_k}(u_k)\left(\prod_{i=1}^{k-1} R^t_{\tilde{a}_i a}(u_i - v)\right)\left(\prod_{j=k-1}^{1} R_{a_j \tilde{a}_k}(\tilde{u}_k - u_j)\right) R^t_{\tilde{a}_k a}(u_k - v) \\
&\qquad \times \left(\prod_{i=1}^{k} R^t_{a_i a}(\tilde{u}_i - v)\right) A_a(v)\left(\prod_{i=k}^{1} R^t_{a_i a}(u_i - v \pm 1)\right)\left(\prod_{i=k}^{1} R^t_{\tilde{a}_i a}(\tilde{u}_i - v \pm 1)\right) \\
&\quad + UWT \\
&= \beta_{\tilde{a}_k a_k}(u_k)\left(\prod_{j=k-1}^{1} R_{a_j \tilde{a}_k}(\tilde{u}_k - u_j)\right)\left(\prod_{i=1}^{k} R^t_{\tilde{a}_i a}(u_i - v)\right) \\
&\qquad \times \left(\prod_{i=1}^{k} R^t_{a_i a}(\tilde{u}_i - v)\right) A_a(v)\left(\prod_{i=k}^{1} R^t_{a_i a}(u_i - v \pm 1)\right)\left(\prod_{i=k}^{1} R^t_{\tilde{a}_i a}(\tilde{u}_i - v \pm 1)\right) \\
&\quad + UWT \\
&= \beta_{\tilde{a}_k a_k}(u_k)\left(\prod_{j=k-1}^{1} R_{a_j \tilde{a}_k}(\tilde{u}_k - u_j)\right) S^{(1)}_a(v; u_1, \ldots, u_k) + UWT
\end{aligned}
$$

as required. ☐

We may apply this result inductively to the creation operator for m excitations $\beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u})$.

Corollary 3.2.18. *The AB exchange relation for multiple excitations has the form*

$$\{p(v)\, A_a(v)\}^v \, \beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u}) = \beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u}) \left\{p(v)\, S_a^{(1)}(v;\boldsymbol{u})\right\}^v + UWT \tag{3.2.67}$$

where $S_a^{(1)}(v;\boldsymbol{u})$ is the nested monodromy matrix for m excitations defined by (3.2.66) and UWT denotes the terms that do not contain $A_a(v)$. ☐

3.2.8 Exchange relations for the nested monodromy matrix

We introduce a vector space $M^{(1)}$, which we call the *nested vacuum sector*, and a matrix $S_a^{(1)}(v;\boldsymbol{w},\boldsymbol{u})$, called the *generalised nested monodromy matrix*, acting on this space, with $\boldsymbol{w} = (w_1, w_2, \ldots, w_m)$ and $\boldsymbol{u} = (u_1, u_2, \ldots, u_m)$ being m-tuples of non-zero complex parameters. We show that $S_a^{(1)}(v;\boldsymbol{w},\boldsymbol{u})$ satisfies the defining relations of the algebra $\mathcal{B}_\rho^{\mathrm{ex}}(n,p)$ in the space $M^{(1)}$. This allows us to identify $S_a^{(1)}(v;\boldsymbol{w},\boldsymbol{u})$ as the monodromy matrix for the residual $\mathcal{B}_\rho^{\mathrm{ex}}(n,p)$-chain, in a suitable sense. The space $M^{(1)}$ is then reinterpreted as the (full) quantum space of this residual chain, which we have studied in Section 3.1.

For each bulk vector space $L(\lambda^{(i)})_{c_i}$ in (3.2.27) denote by $L^0(\lambda^{(i)})_{c_i}$ the subspace consisting of vectors annihilated by the operator $\overline{C}(u)$ of the generating matrix $T(u)$ of $X(\mathfrak{g}_{2n})$, namely

$$L^0(\lambda^{(i)})_{c_i} := \{\zeta \in L(\lambda^{(i)})_{c_i} \,:\, t_{n+k,l}(u)\cdot\zeta = 0 \;\text{ for }\; 1 \le k,l \le n\}. \tag{3.2.68}$$

Lemma 3.2.19. *The space $L^0(\lambda^{(i)})_{c_i}$ is an irreducible lowest weight $Y(\mathfrak{gl}_n)$-module.*

Proof. Relation (3.2.39) implies that $L^0(\lambda^{(i)})_{c_i}$ is stable under the action of $\overline{A}(u)$. Then (3.2.37) allows us to view $L^0(\lambda^{(i)})_{c_i}$ as a $Y(\mathfrak{gl}_n)$-module. Thus we only need to show that $L^0(\lambda^{(i)})_{c_i}$ is an irreducible $Y(\mathfrak{gl}_n)$-module. Let $\eta \in L(\lambda^{(i)})_{c_i}$ be a lowest vector and note that $\eta \in L^0(\lambda^{(i)})_{c_i}$. Set $L := Y(\mathfrak{gl}_n)\,\eta$ and note that $L \subseteq L^0(\lambda^{(i)})_{c_i}$. Since there are no more lowest vectors in $L^0(\lambda^{(i)})_{c_i}$, it follows that $L = L^0(\lambda^{(i)})_{c_i}$. ☐

Introduce a *vacuum sector* M^0 of the full quantum space M by

$$M^0 := L^0(\lambda^{(1)})_{c_1} \otimes \cdots \otimes L^0(\lambda^{(\ell)})_{c_\ell} \otimes V(\mu) \subset M.$$

The Lemma below is an analogue of Lemma 2.2.11.

Lemma 3.2.20. *The operator $C(u)$ of the matrix $S(u)$ acts by zero on the space M^0. Consequently, M^0 is stable under the action of the operator $A(u)$ of the matrix $S(u)$.* ☐

Recall (3.2.65). We define the level-1 *nested vacuum sector* by

$$M^{(1)} := W_{\tilde{\boldsymbol{a}}\boldsymbol{a}} \otimes M^0. \tag{3.2.69}$$

Here an overlap of notation with $M^{(1)}$ defined in Section 3.1.3 is intentional. It will be shown below that $M^{(1)}$ can be viewed as the (full) quantum space for a residual $\mathcal{B}^{\mathrm{ex}}_{\rho}(n,p)$-chain.

Next, we define a generalised nested monodromy matrix which differs from the one in Definition 3.2.16 by an addition m-tuple of complex parameters, $\boldsymbol{w}$. These parameters will play a prominent role in Section 3.2.10.

Definition 3.2.21. *The generalised nested monodromy matrix is defined by*

$$S^{(1)}_a(v;\boldsymbol{w},\boldsymbol{u}) := \left(\prod_{i=1}^{m} R^t_{\tilde{a}_i a}(u_i - v)\right)\left(\prod_{i=1}^{m} R^t_{a_i a}(w_i - v)\right) \times A_a(v)\left(\prod_{i=m}^{1} R^t_{a_i a}(\tilde{w}_i - v \pm 1)\right)\left(\prod_{i=m}^{1} R^t_{\tilde{a}_i a}(\tilde{u}_i - v \pm 1)\right). \tag{3.2.70}$$

Matrix $S^{(1)}_a(v;\boldsymbol{u})$ defined by (3.2.66) is recovered by setting $w_i = \tilde{u}_i$. It will be useful to know that (3.2.35) allows us to rewrite (3.2.70) as

$$S^{(1)}_a(v;\boldsymbol{w},\boldsymbol{u}) = \left(\prod_{i=1}^{m} R^t_{\tilde{a}_i a}(u_i - v)\right)\left(\prod_{i=1}^{m} R^t_{a_i a}(w_i - v)\right) \times A_a(v)\left(\prod_{i=1}^{m} R^t_{a_i a}(w_i + v + \rho)\right)^{-1}\left(\prod_{i=1}^{m} R^t_{\tilde{a}_i a}(u_i + v + \rho)\right)^{-1}. \tag{3.2.71}$$

Set $r = 0$ for types CI, DII and CD0, and $r = n - \frac{p}{2}$ for types DI and CII.

Proposition 3.2.22. *The mapping*

$$\mathcal{B}^{\mathrm{ex}}_{\rho}(n,p) \to \mathrm{End}(M^{(1)}) \otimes X_{\rho}(\mathfrak{g}_{2n},\mathfrak{g}^{\theta}_{2n})^{tw}, \qquad B^{\circ}_a(v) \mapsto S^{(1)}_a(v;\boldsymbol{w},\boldsymbol{u}) \tag{3.2.72}$$

equips the space $M^{(1)}$ with the structure of a lowest weight $\mathcal{B}^{\mathrm{ex}}_{\rho}(n,p)$-module with lowest weight given by

$$\widetilde{\gamma}_j(v;\boldsymbol{w},\boldsymbol{u}) = g(v)\,\widetilde{\mu}^{\circ}_j(v)\left(\prod_{i=1}^{\ell} \lambda^{(i)}_j(v - \tfrac{\kappa}{2})\,\bar{\lambda}^{(i)}_j(\tilde{v} - \tfrac{\kappa}{2})\right), \tag{3.2.73}$$

$$\widetilde{\gamma}_n(v;\boldsymbol{w},\boldsymbol{u}) = g(v)\,\widetilde{\mu}^{\circ}_n(v)\left(\prod_{i=1}^{m} \frac{v-u_i+1}{v-u_i}\cdot\frac{v-w_i+1}{v-w_i}\cdot\frac{v-\tilde{w}_i\mp1+1}{v-\tilde{w}_i\mp1}\cdot\frac{v-\tilde{u}_i\mp1+1}{v-\tilde{u}_i\mp1}\right) \times\left(\prod_{i=1}^{\ell} \lambda^{(i)}_n(v - \tfrac{\kappa}{2})\,\bar{\lambda}^{(i)}_n(\tilde{v} - \tfrac{\kappa}{2})\right) \tag{3.2.74}$$

for $1 \le j \le n-1$ with $g(v)$ defined by (3.2.12), $\mu_j(v)$, $\mu_n(v)$ defined in Proposition 3.2.8, and $\lambda^{(i)}_j(v - \frac{\kappa}{2})$, $\bar{\lambda}^{(i)}_j(\tilde{v} - \frac{\kappa}{2})$ given by

$$\lambda^{(i)}_j(v) = 1 - \frac{\lambda^{(i)}_j}{v - c_i}, \qquad \bar{\lambda}^{(i)}_j(v) = 1 + \frac{\lambda^{(i)}_j}{v - c_i \mp k_i \pm 1 - \kappa} \tag{3.2.75}$$

for $1 \le j \le n$, where $\lambda^{(i)} = (k_i, 0, \dots, 0)$ in the orthogonal case and $\lambda^{(i)} = (1, \dots, 1, 0, \dots, 0)$, with the number of 1's being k_i, in the symplectic case.

Proof. We start by proving the Proposition in the case $m = 0$. Relation (3.2.42) with Lemma 3.2.20 imply that $A(v)$ satisfies the reflection equation on M^0. That is, for any $\zeta \in M^0$,

$$R_{ab}(v-x)A_a(v)R_{ab}(v+x+\rho)A_b(x)\cdot\zeta = A_b(x)R_{ab}(v+x+\rho)A_a(v)R_{ab}(v-x)\cdot\zeta.$$

The remaining terms, which contain $C(u)$ as the rightmost operator, vanish due to Lemma 3.2.20. It follows that M^0 is a lowest weight $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$-module, with weights obtained from (3.2.6) and Proposition 3.2.6. The $m > 0$ case is then immediate from Proposition 3.2.6 and (3.2.71), as the auxiliary spaces are regarded as dual vector evaluation representations of $Y(\mathfrak{gl}_n)$ with shifts of u_i or w_i for $1 \le i \le m$, and lowest weight vector e_1. □

Proposition 3.2.22 implies that $M^{(1)}$ can be viewed as the (full) quantum space for a residual $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$-chain (since $S^{(1)}_a(v;\boldsymbol{w},\boldsymbol{u})$ satisfies (3.1.2) but not the unitarity relation). We end this section with a lemma which will assist us in finding the explicit expressions of the unwanted terms. Recall that $\check{R}(u) := \frac{u}{u-1}PR(u)$.

Lemma 3.2.23. *The following identities hold:*

$$\begin{aligned}
\check{R}(u)\,e_1\otimes e_1 &= e_1\otimes e_1,\\
\check{R}_{\tilde{a}_i\tilde{a}_{i+1}}(u_{i+1}-u_i)\,s_{kl}(v;\boldsymbol{w},\boldsymbol{u}) &= s_{kl}(v;\boldsymbol{w},\boldsymbol{u}_{i\leftrightarrow i+1})\,\check{R}_{\tilde{a}_i\tilde{a}_{i+1}}(u_{i+1}-u_i),\\
\check{R}_{a_ia_{i+1}}(w_{i+1}-w_i)\,s_{kl}(v;\boldsymbol{w},\boldsymbol{u}) &= s_{kl}(v;\boldsymbol{w}_{i\leftrightarrow i+1};\boldsymbol{u})\,\check{R}_{a_ia_{i+1}}(w_{i+1}-w_i).
\end{aligned}$$

Proof. The first identity follows from the definition of $\check{R}(u)$. To obtain the second identity we need to move $\check{R}_{\tilde{a}_i\tilde{a}_{i+1}}(u_{i+1}-u_i)$ rightward through the products of R-matrices in the definition of $S^{(1)}_a(v;\boldsymbol{w},\boldsymbol{u})$ in (3.2.70). In each product we must use the (braided) Yang-Baxter equation once. For $\check{R}_{\tilde{a}_i\tilde{a}_{i+1}}(u_{i+1}-u_i)$ in the leftmost product,

$$\check{R}_{\tilde{a}_i\tilde{a}_{i+1}}(u_{i+1}-u_i)R^t_{\tilde{a}_ia}(u_i-v)R^t_{\tilde{a}_{i+1}a}(u_{i+1}-v) = R^t_{\tilde{a}_ia}(u_{i+1}-v)R^t_{\tilde{a}_{i+1}a}(u_i-v)\check{R}_{\tilde{a}_i\tilde{a}_{i+1}}(u_{i+1}-u_i),$$

and in the rightmost product,

$$\begin{aligned}
&\check{R}_{\tilde{a}_i\tilde{a}_{i+1}}(u_{i+1}-u_i)R^t_{\tilde{a}_{i+1}a}(\tilde{u}_{i+1}-v\pm1)R^t_{\tilde{a}_ia}(\tilde{u}_i-v\pm1)\\
&\qquad\qquad = R^t_{\tilde{a}_{i+1}a}(\tilde{u}_i-v\pm1)R^t_{\tilde{a}_ia}(\tilde{u}_{i+1}-v\pm1)\check{R}_{\tilde{a}_i\tilde{a}_{i+1}}(u_{i+1}-u_i).
\end{aligned}$$

Applying these identities, we obtain the second identity. The third identity is obtained similarly. □

At this point it will be necessary to separate the orthogonal and symplectic cases—we will find that the method used for the symplectic case results in an eigenvector which would be identically equal to zero in the orthogonal case. Thus, we present first the symplectic case in Section 3.2.9, and give the appropriate modifications, following [DVK87], in Section 3.2.10.

3.2.9 Transfer matrix and Bethe vectors for a $X_\rho(\mathfrak{sp}_{2n},\mathfrak{sp}_{2n}^\theta)^{tw}$-chain

Introduce the transfer matrix acting on the quantum space M defined in (3.2.27).

Definition 3.2.24. *The transfer matrix $\tau(v) \in \mathrm{End}(M)[v, v^{-1}]$ is the representative of*

$$\frac{\operatorname{tr} S(v)}{2v - 2\kappa - \rho}$$

on the space M.

From arguments given in [Sk88] (see also Section 2.2 in [Vl15]) the reflection equation (3.2.10) implies that transfer matrices commute,

$$[\tau(u), \tau(v)] = 0.$$

Lemma 3.2.11 allows us to deduce the following symmetry properties of the transfer matrix.

Corollary 3.2.25. *The transfer matrix satisfies the following:*

$$\tau(\tilde v) = \tau(v) = \{p(v) \operatorname{tr} A(v)\}^v.$$

Recall the generalised nested monodromy matrix $S_a^{(1)}(v;\boldsymbol{w},\boldsymbol{u})$ defined in Definition 3.2.21, and the nested vacuum sector $M^{(1)}$ from (3.2.69). By Proposition 3.2.22 we regard $M^{(1)}$ as the (full) quantum space of a residual $\mathcal{B}_\rho^{\mathrm{ex}}(n,p)$-chain. Let $\Phi^{(1)}(\boldsymbol{u}^{(1\ldots n-1)};\boldsymbol{w},\boldsymbol{u})$ denote the level-1 Bethe vector constructed from $S_a^{(1)}(v;\boldsymbol{w},\boldsymbol{u})$ according to Definition 3.1.16.

Lemma 3.2.26. *The level-1 Bethe vector satisfies*

$$\begin{aligned}\check R_{\tilde a_i \tilde a_{i+1}}(w_i - w_{i+1})\,\Phi^{(1)}(\boldsymbol{u}^{(1\ldots n-1)};\boldsymbol{w},\boldsymbol{u}) &= \Phi^{(1)}(\boldsymbol{u}^{(1\ldots n-1)};\boldsymbol{w}_{i\leftrightarrow i+1};\boldsymbol{u}),\\ \check R_{a_i a_{i+1}}(u_i - u_{i+1})\,\Phi^{(1)}(\boldsymbol{u}^{(1\ldots n-1)};\boldsymbol{w},\boldsymbol{u}) &= \Phi^{(1)}(\boldsymbol{u}^{(1\ldots n-1)};\boldsymbol{w},\boldsymbol{u}_{i\leftrightarrow i+1}).\end{aligned}$$

Proof. The level-1 Bethe vector is constructed from a linear combination of products of matrix elements $s_{kl}(v;\boldsymbol{w},\boldsymbol{u})$ of the generalised nested monodromy matrix acting on the highest weight vector. The result is therefore immediate from Lemma 3.2.23. □

Recall the creation operator $\beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u})$ for m excitations from Definition 3.2.14. In what follows we will use the notation $u_i^{(n)} := u_i - \frac{\kappa}{2} = u_i - \frac{n+1}{2}$ and $m_n := m$. Additionally, we will use $\boldsymbol{u} + a$ to mean $(u_1 + a, \ldots, u_m + a)$.

Definition 3.2.27. *The (top-level) symplectic Bethe vector is defined by*

$$\begin{aligned}\Psi(\boldsymbol{u}^{(1\ldots n)}) &:= \beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u}^{(n)} + \tfrac{\kappa}{2}) \cdot \Phi^{(1)}(\boldsymbol{u}^{(1\ldots n-1)}; \tfrac{\kappa}{2} - \boldsymbol{u}^{(n)} - \rho, \boldsymbol{u}^{(n)} + \tfrac{\kappa}{2})\\ &= \beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u}) \cdot \Phi^{(1)}(\boldsymbol{u}^{(1\ldots n-1)}; \tilde{\boldsymbol{u}}, \boldsymbol{u}),\end{aligned}$$

where $\tilde{\boldsymbol{u}} = (\tilde u_1, \ldots, \tilde u_m)$ with $\tilde u_i = \kappa - u_i - \rho$.

As with the $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$ case, $\mathfrak{S}_{\boldsymbol{m}} := \mathfrak{S}_{m_1} \times \cdots \times \mathfrak{S}_{m_{n-1}} \times \mathfrak{S}_{m_n}$ acts on the symplectic Bethe vector by reordering parameters. The invariance of the Bethe vector under this action can then be shown by combining Lemma 3.2.15 and Lemma 3.2.26.

Corollary 3.2.28. *The symplectic Bethe vector is invariant under the action of* $\mathfrak{S}_{\boldsymbol{m}}$. □

Recall the notation $\Lambda^{\pm}(v;\boldsymbol{u}^{(k)})$ in (3.1.29) and in addition define

$$\Lambda^{+2}(v,\boldsymbol{u}^{(n)}) := \prod_{i=1}^{m_n} \frac{(v+u_i^{(n)}+2+\rho)(v-u_i^{(n)}+2)}{(v+u_i^{(n)}+\rho)(v-u_i^{(n)})}.$$

The Theorem below is our first main result.

Theorem 3.2.29. *The symplectic Bethe vector* $\Psi(\boldsymbol{u}^{(1\dots n)})$ *is an eigenvector of the transfer matrix* $\tau(v)$ *with eigenvalue*

$$\Lambda(v;\boldsymbol{u}^{(1\dots n)}) := \big\{p(v)\,\Lambda^{(1)}\big(v;\boldsymbol{u}^{(1\dots n)}\big)\big\}^{v} \tag{3.2.76}$$

where

$$\begin{aligned}\Lambda^{(1)}\big(v;\boldsymbol{u}^{(1\dots n)}\big) := {}& \frac{2v-n+\rho}{2v-1+\rho}\,\Lambda^{+}\big(v-\tfrac{1}{2},\boldsymbol{u}^{(1)}\big)\,\frac{\widetilde{\gamma}_1(v)}{2v+\rho} \\ &+\sum_{i=2}^{n-1}\frac{2v-n+\rho}{2v-i+\rho}\,\Lambda^{-}\big(v-\tfrac{i-1}{2},\boldsymbol{u}^{(i-1)}\big)\,\Lambda^{+}\big(v-\tfrac{i}{2},\boldsymbol{u}^{(i)}\big)\,\frac{\widetilde{\gamma}_i(v)}{2v-i+1+\rho} \\ &+\Lambda^{-}\big(v-\tfrac{n-1}{2},\boldsymbol{u}^{(n-1)}\big)\,\Lambda^{+2}\big(v-\tfrac{\kappa}{2},\boldsymbol{u}^{(n)}\big)\,\frac{\widetilde{\gamma}_n(v)}{2v-n+1+\rho}\end{aligned}$$

and

$$\widetilde{\gamma}_j(v) = g(v)\,\widetilde{\mu}_j^{\circ}(v)\prod_{i=1}^{\ell}\lambda_j^{(i)}(v-\tfrac{\kappa}{2})\prod_{i=1}^{\ell}\lambda_j^{\prime(i)}(\tilde{v}-\tfrac{\kappa}{2})$$

for $1\le j\le n$*, provided*

$$\operatorname*{Res}_{v\to u_j^{(i)}}\Lambda(v+\tfrac{i}{2};\boldsymbol{u}^{(1\dots n)}) = 0 \qquad \textit{and} \qquad \operatorname*{Res}_{v\to u_k^{(n)}}\Lambda(v+\tfrac{\kappa}{2};\boldsymbol{u}^{(1\dots n)}) = 0 \tag{3.2.77}$$

for $1\le j\le m_i$, $1\le i\le n-1$ *and* $1\le k\le m_n$.

Remark 3.2.30. The equations (3.2.77) are Bethe equations for a $X_\rho(\mathfrak{sp}_{2n},\mathfrak{sp}_{2n}^{\theta})^{tw}$-chain. Their explicit form for $u_j^{(i)}$ with $1\le i\le n-2$ is the same as in (3.1.39). Those for $u_j^{(n-1)}$ receive an additional factor due to the top-level excitations,

$$
\begin{aligned}
&\frac{\widetilde{\gamma}_{n-1}\big(u_j^{(n-1)}+\frac{n-1}{2}\big)}{\widetilde{\gamma}_{n}\big(u_j^{(n-1)}+\frac{n-1}{2}\big)} \prod_{\substack{i=1\\ i\neq j}}^{m_{n-1}} \frac{(u_j^{(n-1)}-u_i^{(n-1)}+1)(u_j^{(n-1)}+u_i^{(n-1)}+1+\rho)}{(u_j^{(n-1)}-u_i^{(n-1)}-1)(u_j^{(n-1)}+u_i^{(n-1)}-1+\rho)} \\
&\quad= \prod_{i=1}^{m_{n-2}} \frac{(u_j^{(n-1)}-u_i^{(n-2)}+\frac{1}{2})(u_j^{(n-1)}+u_i^{(n-2)}+\frac{1}{2}+\rho)}{(u_j^{(n-1)}-u_i^{(n-2)}-\frac{1}{2})(u_j^{(n-1)}+u_i^{(n-2)}-\frac{1}{2}+\rho)} \\
&\quad\quad\times \prod_{i=1}^{m_n} \frac{(u_j^{(n-1)}-u_i^{(n)}+1)(u_j^{(n-1)}+u_i^{(n)}+1+\rho)}{(u_j^{(n-1)}-u_i^{(n)}-1)(u_j^{(n-1)}+u_i^{(n)}-1+\rho)}.
\end{aligned}
\tag{3.2.78}
$$

The top-level Bethe equations, for $u_j^{(n)}$, are

$$
\begin{aligned}
&\frac{\widetilde{\gamma}_n(u_j^{(n)}+\frac{\kappa}{2})}{\widetilde{\gamma}_n(\frac{\kappa}{2}-u_j^{(n)}-\rho)} \prod_{\substack{i=1\\ i\neq j}}^{m_n} \frac{(u_j^{(n)}-u_i^{(n)}+2)(u_j^{(n)}+u_i^{(n)}+2+\rho)}{(u_j^{(n)}-u_i^{(n)}-2)(u_j^{(n)}+u_i^{(n)}-2+\rho)} \\
&\quad= \prod_{i=1}^{m_{n-1}} \frac{(u_j^{(n)}-u_i^{(n-1)}+1)(u_j^{(n)}+u_i^{(n-1)}+1+\rho)}{(u_j^{(n)}-u_i^{(n-1)}-1)(u_j^{(n)}+u_i^{(n-1)}-1+\rho)}.
\end{aligned}
\tag{3.2.79}
$$

Proof of Theorem 3.2.29. In order to prove the theorem, it will be necessary to calculate an expression for the unwanted terms. As such, we will first expand on the exchange relations of the twisted Yangian, studied in Section 3.2.7. Recall Corollary 3.2.18,

$$\{p(v)\,A_a(v)\}^v \beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u}) = \beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u})\,\{p(v)\,S_a^{(1)}(v;\boldsymbol{u})\}^v + UWT.$$

Let X_B denote the subalgebra of $X_\rho(\mathfrak{sp}_{2n},\mathfrak{sp}_{2n}^\theta)^{tw}$ generated by elements of the B block matrix, i.e. $s_{i,n+j}^{(k)}$ with $1\le i,j\le n$, $k\ge 1$. The closure of X_B is guaranteed by (3.2.45). Then, considering repeated applications of Lemma 3.2.12, it is possible to write UWT above such that, in each term, elements of the X_B subalgebra appear to the left of the expression. That is, there exist $B_{ij}^{\pm,k}\in W^*_{\tilde{\boldsymbol{a}}\boldsymbol{a}}\otimes X_B((v^{-1}))$ such that

$$
\begin{aligned}
\mathrm{tr}_a\{p(v)\,A_a(v)\}^v\beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u}) &= \beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u})\;\mathrm{tr}_a\{p(v)\,S_a^{(1)}(v;\boldsymbol{u})\}^v \\
&\quad+\sum_{k=1}^{m}\sum_{i,j=1}^{n}\big(B_{ij}^{+,k}\cancel{a}_{ij}(u_k)+B_{ij}^{-,k}\cancel{a}_{ij}(\tilde{u}_k)\big).
\end{aligned}
$$

Since we will not need the exact form the $B_{ij}^{\pm,k}$, we define the combination

$$U^k(v;\boldsymbol{u}) := \sum_{i,j=1}^{n}\big(B_{ij}^{+,k}\cancel{a}_{ij}(u_k)+B_{ij}^{-,k}\cancel{a}_{ij}(\tilde{u}_k)\big),$$

where we have made explicit the dependence on v and $\boldsymbol{u}$. From Lemma 3.2.13 we obtain an exact expression for the unwanted terms for a single excitation. Applying this to the leftmost creation operator $\beta_{\tilde{a}_1 a_1}(u_1)$, followed by Lemma 3.2.17, we are able to extract an expression for $U^1(v;\boldsymbol{u})$:

$$U^1(v;\boldsymbol{u}) = \frac{1}{p(u_1)}\left\{p(v)\,\frac{\beta_{\tilde{a}_1 a_1}(v)}{u_1 - v}\right\}^v \times \prod_{i=2}^{m}\Big(\beta_{\tilde{a}_i a_i}(u_i)\prod_{j=i-1}^{1} R_{a_j \tilde{a}_i}(-u_j - u_i - \rho)\Big) \operatorname*{Res}_{w\to u_1} \operatorname{tr}_a\left\{p(w)\,S_a^{(1)}(w;\boldsymbol{u})\right\}^w. \tag{3.2.80}$$

From here, to find $U^k(v;\boldsymbol{u})$ for $2 \le k \le m$ we make use of Lemma 3.2.15. Specifically, by repeatedly applying transpositions, we may apply any permutation $\sigma \in \mathfrak{S}_m$ to the parameters $\boldsymbol{u}$. Let $\boldsymbol{u}_\sigma$ denote $(u_{\sigma(1)},\ldots,u_{\sigma(m)})$, and let σ_k denote the cyclic permutation $(k, k+1, \ldots, 1, m, \ldots, k-1)$. Then

$$\beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u}) = \beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u}_{\sigma_k})\,\check{R}_{\boldsymbol{a}}[\sigma_k](\boldsymbol{u})\,\check{R}_{\tilde{\boldsymbol{a}}}[\sigma_k](\tilde{\boldsymbol{u}})$$

where $\check{R}_{\boldsymbol{a}}[\sigma_k](\boldsymbol{u})$ is the product of $\check{R}$ matrices necessary to implement this cyclic permutation,

$$\check{R}_{\boldsymbol{a}}[\sigma_k](\boldsymbol{u}) = \prod_{j=k-1}^{1}\left(\prod_{i=m-1}^{1}\check{R}_{a_i a_{i+1}}(u_j - u_{j+1})\right).$$

With this permuted creation operator, repeating the arguments used to find (3.2.80) yields

$$U^k(v;\boldsymbol{u}) = \frac{1}{p(u_k)}\left\{\frac{p(v)}{u_k - v}\beta_{\tilde{a}_1 a_1}(v)\right\}^v \prod_{i=2}^{m}\Big(\beta_{\tilde{a}_i a_i}(u_{\sigma_k(i)})\prod_{j=i-1}^{1} R_{a_j \tilde{a}_i}(-u_{\sigma_k(j)} - u_{\sigma_k(i)} - \rho)\Big) \times \operatorname*{Res}_{w\to u_k}\left\{p(w)\operatorname{tr}_a S_a^{(1)}(w;\boldsymbol{u}_{\sigma_k})\right\}^w \check{R}_{\boldsymbol{a}}[\sigma_k](\boldsymbol{u})\,\check{R}_{\tilde{\boldsymbol{a}}}[\sigma_k](\tilde{\boldsymbol{u}}), \tag{3.2.81}$$

and therefore a full expression for the unwanted terms,

$$\begin{aligned}\operatorname{tr}_a\{p(v)A_a(v)\}^v\beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u}) &= \beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u})\,\operatorname{tr}_a\{p(v)\,S_a^{(1)}(v;\boldsymbol{u})\}^v \\ &+ \sum_{k=1}^{m}\frac{1}{p(u_k)}\left\{\frac{p(v)}{u_k - v}\beta_{\tilde{a}_1 a_1}(v)\right\}^v \prod_{i=2}^{m}\Big(\beta_{\tilde{a}_i a_i}(u_{\sigma_k(i)})\prod_{j=i-1}^{1} R_{a_j\tilde{a}_i}(-u_{\sigma_k(j)} - u_{\sigma_k(i)} - \rho)\Big) \\ &\quad\times \operatorname*{Res}_{w\to u_k}\operatorname{tr}_a\left\{p(w)S_a^{(1)}(w;\boldsymbol{u}_{\sigma_k})\right\}^w \check{R}_{\boldsymbol{a}}[\sigma_k](\boldsymbol{u})\,\check{R}_{\tilde{\boldsymbol{a}}}[\sigma_k](\tilde{\boldsymbol{u}}).\end{aligned}$$

Acting now with this expression on the level-1 Bethe vector gives the full action for the transfer matrix on the top level Bethe vector with $u_i^{(n)} = u_i - \frac{\kappa}{2}$,

$$\begin{aligned}\tau(v)\cdot\Psi(\boldsymbol{u}^{(1\ldots n)}) &= \beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u})\,\mathrm{tr}_a\{p(v)\,S_a^{(1)}(v;\boldsymbol{u})\}^v\cdot\Phi^{(1)}(\boldsymbol{u}^{(1\ldots n-1)};\tilde{\boldsymbol{u}},\boldsymbol{u})\\ &\quad+\sum_{k=1}^{m}\frac{1}{p(u_k)}\left\{\frac{p(v)}{u_k-v}\beta_{\tilde{a}_1a_1}(v)\right\}^v\prod_{i=2}^{m}\Big(\beta_{\tilde{a}_ia_i}(u_{\sigma_k(i)})\prod_{j=i-1}^{1}R_{a_j\tilde{a}_i}(-u_{\sigma_k(j)}-u_{\sigma_k(i)}-\rho)\Big)\\ &\qquad\times\operatorname*{Res}_{w\to u_k}\,\mathrm{tr}_a\left\{p(w)\,S_a^{(1)}(w;\boldsymbol{u}_{\sigma_k})\right\}^w\check{R}_{\boldsymbol{a}}[\sigma_k](\boldsymbol{u})\,\check{R}_{\tilde{\boldsymbol{a}}}[\sigma_k](\tilde{\boldsymbol{u}})\cdot\Phi^{(1)}(\boldsymbol{u}^{(1\ldots n-1)};\tilde{\boldsymbol{u}},\boldsymbol{u})\\ &=\beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u})\,\mathrm{tr}_a\{p(v)S_a^{(1)}(v;\boldsymbol{u})\}^v\cdot\Phi^{(1)}(\boldsymbol{u}^{(1\ldots n-1)};\tilde{\boldsymbol{u}},\boldsymbol{u})\\ &\quad+\sum_{k=1}^{m}\frac{1}{p(u_k)}\left\{\frac{p(v)}{u_k-v}\beta_{\tilde{a}_1a_1}(v)\right\}^v\prod_{i=2}^{m}\Big(\beta_{\tilde{a}_ia_i}(u_{\sigma_k(i)})\prod_{j=i-1}^{1}R_{a_j\tilde{a}_i}(-u_{\sigma_k(j)}-u_{\sigma_k(i)}-\rho)\Big)\\ &\qquad\times\operatorname*{Res}_{w\to u_k}\,\mathrm{tr}_a\left\{p(w)\,S_a^{(1)}(w;\boldsymbol{u}_{\sigma_k})\right\}^w\cdot\Phi^{(1)}(\boldsymbol{u}^{(1\ldots n-1)};\tilde{\boldsymbol{u}}_{\sigma_k},\boldsymbol{u}_{\sigma_k}).\end{aligned}$$

The last equality follows from Lemma 3.2.26. From the full expression (3.2.76), the condition (3.2.77) for the parameters $u_i^{(j)}$ is equivalent to $\operatorname{Res}_{v\to u_i^{(j)}}\Lambda^{(1)}(v+\frac{j}{2};\boldsymbol{u}^{(1\ldots n)})=0$ with $u_i^{(n)}=u_i-\frac{\kappa}{2}$, as these poles are not present in $\Lambda^{(1)}(\tilde{v}-\frac{j}{2};\boldsymbol{u}^{(1\ldots n)})$. Therefore, from Theorem 3.1.18, using weights from Proposition 3.2.22,

$$\begin{aligned}\tau(v)\cdot\Psi\big(\boldsymbol{u}^{(1\ldots n)}\big) &= \Lambda\big(v;\boldsymbol{u}^{(1\ldots n)}\big)\,\Psi\big(\boldsymbol{u}^{(1\ldots n)}\big)\\ &\quad+\sum_{k=1}^{m}\operatorname*{Res}_{w\to u_k}\Lambda(w;\boldsymbol{u}^{(1\ldots n)}_{\sigma_k^{(n)}})\frac{1}{p(u_k)}\left\{\frac{p(v)}{u_k-v}\beta_{\tilde{a}_1a_1}(v)\right\}^v\\ &\qquad\times\prod_{i=2}^{m}\Big(\beta_{\tilde{a}_ia_i}(u_{\sigma_k(i)})\prod_{j=i-1}^{1}R_{a_j\tilde{a}_i}(-u_{\sigma_k(j)}-u_{\sigma_k(i)}-\rho)\Big)\cdot\Phi^{(1)}\big(\boldsymbol{u}^{(1\ldots n-1)};\tilde{\boldsymbol{u}}_{\sigma_k},\boldsymbol{u}_{\sigma_k}\big),\end{aligned}$$

where $\Lambda(v;\boldsymbol{u}^{(1\ldots n)})=\big\{p(v)\Lambda^{(1)}(v;\boldsymbol{u}^{(1\ldots n)})\big\}^v$ as required. Note that, owing to Corollary 3.2.28, we have $\Lambda(v;\boldsymbol{u}^{(1\ldots n)})=\Lambda(v;\boldsymbol{u}^{(1\ldots n)}_{\sigma^{(n)}})$ for any $\sigma^{(n)}\in\mathfrak{S}_{m_n}$. Therefore, $\Psi(\boldsymbol{u}^{(1\ldots n)})$ is an eigenvector of $\tau(v)$ with eigenvalue $\Lambda(v;\boldsymbol{u}^{(1\ldots n)})$ provided $\operatorname*{Res}_{v\to u_k}\Lambda(v;\boldsymbol{u}^{(1\ldots n)})=0$, or equivalently $\operatorname*{Res}_{v\to u_k^{(n)}}\Lambda(v+\frac{\kappa}{2};\boldsymbol{u}^{(1\ldots n)})=0$, for $1\le k\le m_n$. □

Example 3.2.31. The symplectic Bethe vector with m top-level excitations and $m_1=\ldots=m_{n-1}=0$ is given by

$$\begin{aligned}\Psi(\boldsymbol{u}^{(n)}) &= \big[B(u_1^{(n)}+\tfrac{\kappa}{2})\big]_{n,1}\cdots\big[B(u_m^{(n)}+\tfrac{\kappa}{2})\big]_{n,1}\cdot\xi\\ &=\big[S(u_1^{(n)}+\tfrac{\kappa}{2})\big]_{n,n+1}\cdots\big[S(u_m^{(n)}+\tfrac{\kappa}{2})\big]_{n,n+1}\cdot\xi.\end{aligned}$$

For $m_1=m_n=1$ and $m_2=\ldots=m_{n-1}=0$, the on-shell symplectic Bethe vector, that is, when the parameters satisfy the Bethe equations, takes the form

$$\Psi(u^{(n)}, u^{(n-1)}) = \frac{(u^{(n)} - u^{(n-1)} - 1)(u^{(n)} + u^{(n-1)} + \rho + 1)}{(u^{(n)} - u^{(n-1)})(u^{(n)} + u^{(n-1)} + \rho)}$$
$$\times \Bigg(\big[B(u^{(n)} + \tfrac{\kappa}{2})\big]_{n,1} \big[\hat{A}^{(n-1)}(u^{(n-1)} + \tfrac{1}{2})\big]_{12}$$
$$- \frac{\widetilde{\gamma}_{n-1}(u^{(n-1)} + \frac{n-1}{2})}{(u^{(n)} - u^{(n-1)} - 1)(u^{(n)} + u^{(n-1)} + \rho + 1)}$$
$$\times \Big(\big[B(u^{(n)} + \tfrac{\kappa}{2})\big]_{n,2} + \big[B(u^{(n)} + \tfrac{\kappa}{2})\big]_{n-1,1} \Big) \Bigg) \cdot \xi, \tag{3.2.82}$$

where $\hat{A}^{(n-1)}(v)$ refers to the level-$(n-1)$ nested version of the A operator of $S(v)$ obtained via (3.1.13).

3.2.10 Transfer matrix and Bethe vectors for a $X_\rho(\mathfrak{so}_{2n}, \mathfrak{so}_{2n}^{\theta})^{tw}$-chain

We now focus on the orthogonal case. We define the transfer matrix $\tau(v)$ acting on the quantum space M defined in (3.2.27) in the same way as we did in the symplectic case. However, the definition of the orthogonal Bethe vector will differ from its symplectic counterpart in Definition 3.2.27. Indeed, looking at Proposition 3.2.22, the weights $\widetilde{\gamma}_n(v; \tilde{\boldsymbol{u}}; \boldsymbol{u})$ do not have poles at $v = u_i$, and so making the same ansatz as in the symplectic case would yield Bethe equations that are trivially satisfied. Such an ansatz therefore must be identically equal to zero. To remedy this we use a limiting procedure proposed in [DVK87]. Recall that $\Phi^{(1)}(\boldsymbol{u}^{(1\ldots n-1)}; \boldsymbol{w}; \boldsymbol{u})$ denotes the level-1 Bethe vector constructed from $S_a^{(1)}(v; \boldsymbol{w}; \boldsymbol{u})$ according to Definition 3.1.16.

Definition 3.2.32. *The level-1 orthogonal Bethe vector is defined by*

$$\Phi^{(1)}_{lim}(\boldsymbol{u}^{(1\ldots n-1)}, \tilde{\boldsymbol{u}}; \boldsymbol{\alpha}, \boldsymbol{\beta}) := \lim_{\epsilon \to 0} \Phi^{(1)}(\boldsymbol{u}^{(1\ldots n-2)}, (\boldsymbol{u}^{(n-1)}, \tilde{\boldsymbol{u}} - \tfrac{\kappa}{2} - \epsilon); \tilde{\boldsymbol{u}} - \boldsymbol{\beta}\epsilon; \boldsymbol{u} + \boldsymbol{\alpha}\epsilon).$$

In the above definition, as well as parameters $\boldsymbol{u}^{(1\ldots n-1)}$, the Bethe vector includes m additional excitations at level-$(n-1)$, with parameters $\tilde{u}_i - \frac{\kappa}{2} - \epsilon = \frac{\kappa}{2} - u_i - \rho - \epsilon$. The shift of $\frac{\kappa}{2} = \frac{1}{2}(n-1)$ is simply to account for the parameter shifts in the nested Bethe ansatz for the $\mathcal{B}^{\mathrm{ex}}_\rho(n,p)$-chain. Parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have been introduced to control the limit as $\epsilon \to 0$. These parameters should be thought of as additional Bethe parameters, which will eventually be determined by the Bethe equations. We obtain the same parameter symmetry as Lemma 3.2.26.

Lemma 3.2.33. *The level-1 orthogonal Bethe vector satisfies*

$$\check{R}_{\tilde{a}_i\tilde{a}_{i+1}}(u_{i+1} - u_i)\check{R}_{a_ia_{i+1}}(u_i - u_{i+1})\,\Phi^{(1)}_{lim}(\boldsymbol{u}^{(1\ldots n-1)}, \tilde{\boldsymbol{u}}; \boldsymbol{\alpha}, \boldsymbol{\beta})$$
$$= \Phi^{(1)}_{lim}(\boldsymbol{u}^{(1\ldots n-1)}, \tilde{\boldsymbol{u}}_{i\leftrightarrow i+1}; \boldsymbol{\alpha}_{i\leftrightarrow i+1}, \boldsymbol{\beta}_{i\leftrightarrow i+1}).$$

Proof. We use Lemma 3.1.17 to write

$$\Phi^{(1)}_{lim}(\boldsymbol{u}^{(1\dots n-1)},\tilde{\boldsymbol{u}};\boldsymbol{\alpha},\boldsymbol{\beta})=\Phi^{(1)}_{lim}(\boldsymbol{u}^{(1\dots n-1)},\tilde{\boldsymbol{u}}_{i\leftrightarrow i+1};\boldsymbol{\alpha},\boldsymbol{\beta}).$$

Then

$$\begin{aligned}
&\check{R}_{\tilde{a}_i\tilde{a}_{i+1}}(u_{i+1}-u_i)\check{R}_{a_ia_{i+1}}(u_i-u_{i+1})\lim_{\epsilon\to 0}\Phi^{(1)}(\boldsymbol{u}^{(1\dots n-2)},(\boldsymbol{u}^{(n-1)},\tilde{\boldsymbol{u}}-\tfrac{\kappa}{2}-\epsilon);\tilde{\boldsymbol{u}}-\boldsymbol{\beta}\epsilon;\boldsymbol{u}+\boldsymbol{\alpha}\epsilon)\\
&\quad=\lim_{\epsilon\to 0}\check{R}_{\tilde{a}_i\tilde{a}_{i+1}}(u_{i+1}-u_i+(\beta_i-\beta_{i+1})\epsilon)\check{R}_{a_ia_{i+1}}(u_i-u_{i+1}+(\alpha_i-\alpha_{i+1})\epsilon)\\
&\qquad\qquad\times\Phi^{(1)}(\boldsymbol{u}^{(1\dots n-2)},(\boldsymbol{u}^{(n-1)},\tilde{\boldsymbol{u}}-\tfrac{\kappa}{2}-\epsilon);\tilde{\boldsymbol{u}}-\boldsymbol{\beta}\epsilon;\boldsymbol{u}+\boldsymbol{\alpha}\epsilon)\\
&\quad=\lim_{\epsilon\to 0}\Phi^{(1)}(\boldsymbol{u}^{(1\dots n-2)},(\boldsymbol{u}^{(n-1)},\tilde{\boldsymbol{u}}-\tfrac{\kappa}{2}-\epsilon);\tilde{\boldsymbol{u}}_{i\leftrightarrow i+1}-\boldsymbol{\beta}_{i\leftrightarrow i+1}\epsilon;\boldsymbol{u}_{i\leftrightarrow i+1}+\boldsymbol{\alpha}_{i\leftrightarrow i+1}\epsilon),
\end{aligned}$$

where the last equality follows from Lemma 3.2.23, as in the symplectic case. Then, following up with Lemma 3.1.17 to exchange $\tilde{\boldsymbol{u}}-\frac{\kappa}{2}-\epsilon$ with $\tilde{\boldsymbol{u}}_{i\leftrightarrow i+1}-\frac{\kappa}{2}-\epsilon$, we obtain the desired result. □

Corollary 3.2.34. *The level-1 orthogonal Bethe vector satisfies*

$$\begin{aligned}
&\tau^{(1)}(v;\tilde{\boldsymbol{u}};\boldsymbol{u})\,\Phi^{(1)}_{lim}(\boldsymbol{u}^{(1\dots n-1)},\tilde{\boldsymbol{u}};\boldsymbol{\alpha},\boldsymbol{\beta})\\
&\qquad=\Lambda^{(1)}\big(v;\boldsymbol{u}^{(1\dots n-2)},(\boldsymbol{u}^{(n-1)},\tilde{\boldsymbol{u}}-\tfrac{\kappa}{2});\tilde{\boldsymbol{u}},\boldsymbol{u}\big)\,\Phi^{(1)}_{lim}(\boldsymbol{u}^{(1\dots n-1)},\tilde{\boldsymbol{u}};\boldsymbol{\alpha},\boldsymbol{\beta})
\end{aligned}$$

with

$$\begin{aligned}
\Lambda^{(1)}\big(v;\boldsymbol{u}^{(1\dots n-1)};\boldsymbol{w},\boldsymbol{u}\big)&=\frac{2v-n+\rho}{2v-1+\rho}\,\Lambda^{+}\big(v-\tfrac{1}{2},\boldsymbol{u}^{(1)}\big)\,\frac{\widetilde{\gamma}_1(v)}{2v+\rho}\\
&+\sum_{i=2}^{n-1}\frac{2v-n+\rho}{2v-i+\rho}\,\Lambda^{-}\big(v-\tfrac{i-1}{2},\boldsymbol{u}^{(i-1)}\big)\,\Lambda^{+}\big(v-\tfrac{i}{2},\boldsymbol{u}^{(i)}\big)\,\frac{\widetilde{\gamma}_i(v)}{2v-i+1+\rho}\\
&+\Lambda^{-}\big(v-\tfrac{n-1}{2},\boldsymbol{u}^{(n-1)}\big)\Lambda^{+}\big(v-\tfrac{n}{2},\boldsymbol{w}-\tfrac{n}{2}\big)\Lambda^{+}\big(v-\tfrac{n}{2},\boldsymbol{u}-\tfrac{n}{2}\big)\,\frac{\widetilde{\gamma}_n(v)}{2v-n+1+\rho}
\end{aligned}\tag{3.2.83}$$

provided

$$\operatorname*{Res}_{v\to u^{(i)}_j}\Lambda^{(1)}\big(v+\tfrac{i}{2};\boldsymbol{u}^{(1\dots n-2)},(\boldsymbol{u}^{(n-1)},\tilde{\boldsymbol{u}}-\tfrac{\kappa}{2});\tilde{\boldsymbol{u}},\boldsymbol{u}\big)=0\quad\textit{for}\quad 1\le j\le m_i,\ \ 1\le i\le n-1,\tag{3.2.84}$$

$$\lim_{\epsilon\to 0}\operatorname*{Res}_{v\to\tilde{u}_j-\epsilon}\Lambda^{(1)}\big(v;\boldsymbol{u}^{(1\dots n-2)},(\boldsymbol{u}^{(n-1)},\tilde{\boldsymbol{u}}-\tfrac{\kappa}{2}-\epsilon);\tilde{\boldsymbol{u}}-\boldsymbol{\beta}\epsilon,\boldsymbol{u}+\boldsymbol{\alpha}\epsilon\big)=0\quad\textit{for}\quad 1\le j\le m.\tag{3.2.85}$$

Proof. By Theorem 3.1.18 and Proposition 3.2.22, vector $\Phi^{(1)}(\boldsymbol{u}^{(1\dots n-2)},(\boldsymbol{u}^{(n-1)},\tilde{\boldsymbol{u}}-\frac{\kappa}{2}-\epsilon);\tilde{\boldsymbol{u}}-\boldsymbol{\beta}\epsilon;\boldsymbol{u}+\boldsymbol{\alpha}\epsilon)$ is an eigenvector of the nested transfer matrix $\tau^{(1)}(v;\tilde{\boldsymbol{u}}-\boldsymbol{\beta}\epsilon;\boldsymbol{u}+\boldsymbol{\alpha}\epsilon)$ with eigenvalue $\Lambda^{(1)}=\Lambda^{(1)}(v;\boldsymbol{u}^{(1\dots n-2)},\ (\boldsymbol{u}^{(n-1)},\tilde{\boldsymbol{u}}-\frac{\kappa}{2}-\epsilon);\tilde{\boldsymbol{u}}-\boldsymbol{\beta}\epsilon;\boldsymbol{u}+\boldsymbol{\alpha}\epsilon)$ provided $\operatorname{Res}_{v\to u^{(i)}_j+\frac{i}{2}}\Lambda^{(1)}=0$ for $1\le j\le m_i$, $1\le i\le n-1$ and $\operatorname{Res}_{v\to\tilde{u}_j-\frac{\kappa}{2}-\epsilon+\frac{n-1}{2}}\Lambda^{(1)}=0$ for $1\le j\le m$. Taking the $\epsilon\to 0$ limit gives the wanted result. □

Direct evaluation of the residue and the limit in (3.2.85) yield the following Bethe equations for

$1 \le j \le m$,

$$\frac{2u_j - n + \rho + 2}{2u_j - n + \rho}\frac{\widetilde{\gamma}_{n-1}(\tilde{u}_j)}{\widetilde{\gamma}_n(\tilde{u}_j)} = -\frac{1-\alpha_j}{1-\beta_j}\frac{1}{\Lambda^+(u_j - \frac{n}{2}, \boldsymbol{u}^{(n-2)})}\frac{\Lambda^+(u_j - \frac{n-1}{2}, \boldsymbol{u}^{(n-1)})}{\Lambda^-(u_j - \frac{n-1}{2}, \boldsymbol{u}^{(n-1)})}.$$

For any collection of Bethe roots $\boldsymbol{u}^{(1\ldots n-1)}$, the above equations can be thought to constrain $\boldsymbol{\alpha}$ in terms of $\boldsymbol{\beta}$. With this perspective, for any m-tuple $\boldsymbol{u}$, as the equations depend on α_j and β_j only through the combination $(1-\alpha_j)/(1-\beta_j)$, there is a 1-parameter family of eigenvectors of the nested transfer matrix with the same eigenvalue. We conclude that any choice of $\boldsymbol{\beta}$ must give the same nested Bethe vector. In particular, there will be two choices of interest:

$$\alpha_j = 0, \quad \beta_j = \delta_j \qquad \text{and} \qquad \alpha_j = 1 - \frac{1}{1-\delta_j} =: \hat{\delta}_j, \quad \beta_j = 0,$$

with eigenvector

$$\Phi^{(1)}_{lim}(\boldsymbol{u}^{(1\ldots n-1)}, \tilde{\boldsymbol{u}}; \boldsymbol{0}, \boldsymbol{\delta}) = \Phi^{(1)}_{lim}(\boldsymbol{u}^{(1\ldots n-1)}, \tilde{\boldsymbol{u}}; \hat{\boldsymbol{\delta}}, \boldsymbol{0}). \tag{3.2.86}$$

Note that this equality has only been shown to hold "on-shell", i.e. when the $\boldsymbol{u}^{(1\ldots n-1)}$ satisfy Bethe equations. We are now ready to define the top-level orhtogonal Bethe vector. In what follows, we will write $u_i^{(n)} := u_i - \frac{\kappa}{2} = u_i - \frac{n-1}{2}$, $\bar{v} := -v - \rho$ and $m_n := m$.

Definition 3.2.35. *The (top-level) orthogonal Bethe vector is*

$$\begin{aligned}\Psi(\boldsymbol{u}^{(1\ldots n)}; \boldsymbol{\delta}) &:= \beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u}) \cdot \Phi^{(1)}_{lim}(\boldsymbol{u}^{(1\ldots n-1)}, \tilde{\boldsymbol{u}}; \boldsymbol{0}, \boldsymbol{\delta}) \\ &= \beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u}^{(n)} + \tfrac{\kappa}{2}) \cdot \Phi^{(1)}_{lim}(\boldsymbol{u}^{(1\ldots n-1)}, \bar{\boldsymbol{u}}^{(n)} + \tfrac{\kappa}{2}; \boldsymbol{0}, \boldsymbol{\delta})\end{aligned}$$

with δ_j defined by, for $1 \le j \le m_n$,

$$\delta_j := 1 + \frac{2u_j^{(n)} + \rho - 1}{2u_j^{(n)} + \rho + 1}\frac{\widetilde{\gamma}_n(\bar{u}_j^{(n)} + \frac{\kappa}{2})}{\widetilde{\gamma}_{n-1}(\bar{u}_j^{(n)} + \frac{\kappa}{2})}\frac{1}{\Lambda^+(u_j^{(n)} - \frac{1}{2}, \boldsymbol{u}^{(n-2)})}\frac{\Lambda^+(u_j^{(n)}, \boldsymbol{u}^{(n-1)})}{\Lambda^-(u_j^{(n)}, \boldsymbol{u}^{(n-1)})}. \tag{3.2.87}$$

We now have an $\mathfrak{S}_{\boldsymbol{m}} := \mathfrak{S}_{m_1} \times \cdots \times \mathfrak{S}_{m_{n-1}} \times \mathfrak{S}_{m_n}$ action on the orthogonal Bethe vector by reordering parameters. The invariance of the Bethe vector under this action can then be shown by combining Lemma 3.2.15 and Lemma 3.2.33.

Corollary 3.2.36. *The orthogonal Bethe vector is invariant under the action of $\mathfrak{S}_{\boldsymbol{m}}$.* □

The Theorem below is our second main result. Recall (3.2.83).

Theorem 3.2.37. *The orthogonal Bethe vector $\Psi(\boldsymbol{u}^{(1\ldots n)}; \boldsymbol{\delta})$ is an eigenvector of the transfer matrix $\tau(v)$ with eigenvalue*

$$\Lambda(v; \boldsymbol{u}^{(1\ldots n)}) := \big\{p(v)\Lambda^{(1)}\big(v; \boldsymbol{u}^{(1\ldots n-2)}, (\boldsymbol{u}^{(n-1)}, \bar{\boldsymbol{u}}^{(n)}); \bar{\boldsymbol{u}}^{(n)} + \tfrac{\kappa}{2}, \boldsymbol{u}^{(n)} + \tfrac{\kappa}{2}\big)\big\}^v \tag{3.2.88}$$

provided

$$\operatorname*{Res}_{v\to u_j^{(i)}} \Lambda(v+\tfrac{i}{2};\boldsymbol{u}^{(1\dots n)}) = 0 \tag{3.2.89}$$

for $1\le j\le m_i$, $1\le i\le n-2$, *and*

$$\frac{\widetilde{\gamma}_{n-1}(u_j^{(n-1)}+\frac{\kappa}{2})}{\widetilde{\gamma}_n(u_j^{(n-1)}+\frac{\kappa}{2})}\prod_{\substack{i=1\\ i\neq j}}^{m_{n-1}}\frac{(u_j^{(n-1)}-u_i^{(n-1)}+1)(u_j^{(n-1)}+u_i^{(n-1)}+\rho+1)}{(u_j^{(n-1)}-u_i^{(n-1)}-1)(u_j^{(n-1)}+u_i^{(n-1)}+\rho-1)} = \frac{1}{\Lambda^-(u_j^{(n-1)}+\frac{1}{2},\boldsymbol{u}^{(n-2)})} \tag{3.2.90}$$

for $1\le j\le m_{n-1}$, *and*

$$\frac{\widetilde{\gamma}_n(u_j^{(n)}+\frac{\kappa}{2})}{\widetilde{\gamma}_{n-1}(\bar{u}_j^{(n)}+\frac{\kappa}{2})}\prod_{\substack{i=1\\ i\neq j}}^{m_n}\frac{(u_j^{(n)}-u_i^{(n)}+1)(u_j^{(n)}+u_i^{(n)}+\rho+1)}{(u_j^{(n)}-u_i^{(n)}-1)(u_j^{(n)}+u_i^{(n)}+\rho-1)} = \frac{1}{\Lambda^-(u_j^{(n)}+\frac{1}{2},\boldsymbol{u}^{(n-2)})} \tag{3.2.91}$$

for $1\le j\le m_n$.

Remark 3.2.38. The equations (3.2.89–3.2.91) are Bethe equations for a $X_\rho(\mathfrak{so}_{2n},\mathfrak{so}_{2n}^\theta)^{tw}$-chain. Their explicit form for $u_j^{(i)}$ with $1\le i\le n-3$ and $i=n-1$ is the same as in (3.1.39). For $i=n-2$ there is an additional factor, corresponding to the extra excitations at level $n-1$,

$$\begin{aligned}
&\frac{\widetilde{\gamma}_{n-2}\big(u_j^{(n-2)}+\frac{n-2}{2}\big)}{\widetilde{\gamma}_{n-1}\big(u_j^{(n-2)}+\frac{n-2}{2}\big)}\prod_{\substack{i=1\\ i\neq j}}^{m_{n-2}}\frac{(u_j^{(n-2)}-u_i^{(n-2)}+1)(u_j^{(n-2)}+u_i^{(n-2)}+1+\rho)}{(u_j^{(n-2)}-u_i^{(n-2)}-1)(u_j^{(n-2)}+u_i^{(n-2)}-1+\rho)}\\
&\quad=\prod_{i=1}^{m_{n-3}}\frac{(u_j^{(n-2)}-u_i^{(n-3)}+\frac{1}{2})(u_j^{(n-2)}+u_i^{(n-3)}+\frac{1}{2}+\rho)}{(u_j^{(n-2)}-u_i^{(n-3)}-\frac{1}{2})(u_j^{(n-2)}+u_i^{(n-3)}-\frac{1}{2}+\rho)}\\
&\qquad\times\prod_{i=1}^{m_{n-1}}\frac{(u_j^{(n-2)}-u_i^{(n-1)}+\frac{1}{2})(u_j^{(n-2)}+u_i^{(n-1)}+\frac{1}{2}+\rho)}{(u_j^{(n-2)}-u_i^{(n-1)}-\frac{1}{2})(u_j^{(n-2)}+u_i^{(n-1)}-\frac{1}{2}+\rho)}\\
&\qquad\times\prod_{i=1}^{m_n}\frac{(u_j^{(n-2)}-u_i^{(n)}+\frac{1}{2})(u_j^{(n-2)}+u_i^{(n)}+\frac{1}{2}+\rho)}{(u_j^{(n-2)}-u_i^{(n)}-\frac{1}{2})(u_j^{(n-2)}+u_i^{(n)}-\frac{1}{2}+\rho)}.
\end{aligned} \tag{3.2.92}$$

The sets of parameters $\boldsymbol{u}^{(n-1)}$ and $\boldsymbol{u}^{(n)}$ correspond to the two branching Dynkin nodes of $\mathfrak{so}_{2n}$, and are often denoted $\boldsymbol{u}^{(+)}$ and $\boldsymbol{u}^{(-)}$.

Remark 3.2.39. For $n=2$, the Bethe equations (3.2.90) and (3.2.91) decouple into two sets of Bethe equations for open $\mathfrak{sl}_2$ spin chains, and can be solved separately. This is consistent with the isomorphism $\mathfrak{so}_4\cong\mathfrak{sl}_2\oplus\mathfrak{sl}_2$. Similarly, for $n=3$, the isomorphism $\mathfrak{so}_6\cong\mathfrak{sl}_4$ is borne out in the Bethe equations (3.2.90), (3.2.91) and (3.2.92).

Proof of Theorem 3.2.37. The calculation of unwanted terms is identical to the symplectic case. In particular, using Lemma 3.2.33 we find

$$
\begin{aligned}
\tau(v)\cdot\Psi(\boldsymbol{u}^{(1\ldots n)};\boldsymbol{\delta}) = \beta_{\tilde{\boldsymbol{a}}\boldsymbol{a}}(\boldsymbol{u})\,\{p(v)\,\tau^{(1)}(v;\tilde{\boldsymbol{u}};\boldsymbol{u})\}^{v}\cdot\Phi^{(1)}_{lim}(\boldsymbol{u}^{(1\ldots n-1)},\tilde{\boldsymbol{u}};\boldsymbol{0},\boldsymbol{\delta}) \\
+\sum_{j=1}^{m_n}\frac{1}{p(u_j)}\left\{\frac{p(v)}{u_j-v}\,\beta_{\tilde{a}_1a_1}(v)\right\}^{v} \\
\times\prod_{i=2}^{m_n}\Big(\beta_{\tilde{a}_ia_i}(u_{\sigma_j(i)})\prod_{k=i-1}^{1}R_{a_k\tilde{a}_i}(-u_{\sigma_j(k)}-u_{\sigma_j(i)}-\rho)\Big) \\
\times\operatorname*{Res}_{w\to u_j}\{p(w)\,\tau^{(1)}(w;\tilde{\boldsymbol{u}}_{\sigma_j};\boldsymbol{u}_{\sigma_j})\}^{w}\cdot\Phi^{(1)}_{lim}(\boldsymbol{u}^{(1\ldots n-1)},\tilde{\boldsymbol{u}}_{\sigma_j};\boldsymbol{0},\boldsymbol{\delta}_{\sigma_j}).
\end{aligned}
$$

Recall notation $u_j^{(n)} = u-\frac{\kappa}{2}$. Corollary 3.2.34 applied to the wanted term together with the identity

$$
\operatorname*{Res}_{v\to u_j^{(i)}}\Lambda^{(1)}\big(\tilde{v}-\tfrac{i}{2};\boldsymbol{u}^{(1\ldots n-2)},(\boldsymbol{u}^{(n-1)},\bar{\boldsymbol{u}}^{(n)});\bar{\boldsymbol{u}}^{(n)}+\tfrac{\kappa}{2},\boldsymbol{u}^{(n)}+\tfrac{\kappa}{2}\big)=0
$$

for $1\le j\le m_i$ and $1\le i\le n-2$ yields (3.2.88) and (3.2.89). The above identity does not hold for $i=n-1$. Thus the Bethe equations (3.2.90) for $u_j^{(n-1)}$ are obtained by evaluating directly

$$
\operatorname*{Res}_{v\to u_j^{(n-1)}}\Lambda^{(1)}\big(v-\tfrac{n-1}{2};\boldsymbol{u}^{(1\ldots n-2)},(\boldsymbol{u}^{(n-1)},\bar{\boldsymbol{u}}^{(n)});\bar{\boldsymbol{u}}^{(n)}+\tfrac{\kappa}{2},\boldsymbol{u}^{(n)}+\tfrac{\kappa}{2}\big)=0
$$

and the help of

$$
\Lambda^{-}\big(v,\bar{\boldsymbol{u}}^{(n)}\big)\,\Lambda^{+}\big(v-\tfrac12,\bar{\boldsymbol{u}}^{(n)}-\tfrac12\big)\Lambda^{+}\big(v-\tfrac12,\boldsymbol{u}^{(n)}-\tfrac12\big)=\Lambda^{+}\big(v,\bar{\boldsymbol{u}}^{(n)}\big).
$$

The top-level Bethe equations (3.2.91) for $u_j^{(n)}$ are obtained from equating to zero the unwanted terms. However, some care must be taken so as not to exchange the order of the residue and limit. Using the same arguments as in the proof of Corollary 3.2.34 and assuming (3.2.89) and (3.2.90), so that (3.2.86) holds, we write

$$
\begin{aligned}
&\operatorname*{Res}_{w\to u_j}\{p(w)\,\tau^{(1)}(w;\tilde{\boldsymbol{u}}_{\sigma_j};\boldsymbol{u}_{\sigma_j})\}^{w}\cdot\Phi^{(1)}_{lim}(\boldsymbol{u}^{(1\ldots n-1)},\tilde{\boldsymbol{u}}_{\sigma_j};\boldsymbol{0},\boldsymbol{\delta}_{\sigma_j}) \\
&\quad=\lim_{\epsilon\to0}\operatorname*{Res}_{w\to u_j}\Bigg(p(w)\,\tau^{(1)}(w;\tilde{\boldsymbol{u}}_{\sigma_j}-\boldsymbol{\delta}_{\sigma_j}\epsilon;\boldsymbol{u}_{\sigma_j}) \\
&\qquad\qquad\times\Phi^{(1)}(\boldsymbol{u}^{(1\ldots n-2)},(\boldsymbol{u}^{(n-1)},\tilde{\boldsymbol{u}}_{\sigma_j}-\tfrac{\kappa}{2}-\epsilon);\tilde{\boldsymbol{u}}_{\sigma_j}-\boldsymbol{\delta}_{\sigma_j}\epsilon;\boldsymbol{u}_{\sigma_j}) \\
&\qquad\quad+p(\tilde{w})\,\tau^{(1)}(\tilde{w};\tilde{\boldsymbol{u}}_{\sigma_j};\boldsymbol{u}_{\sigma_j}+\hat{\boldsymbol{\delta}}_{\sigma_j}\epsilon) \\
&\qquad\qquad\times\Phi^{(1)}(\boldsymbol{u}^{(1\ldots n-2)},(\boldsymbol{u}^{(n-1)},\tilde{\boldsymbol{u}}_{\sigma_j}-\tfrac{\kappa}{2}-\epsilon);\tilde{\boldsymbol{u}}_{\sigma_j};\boldsymbol{u}_{\sigma_j}+\hat{\boldsymbol{\delta}}_{\sigma_j}\epsilon)\Bigg).
\end{aligned}
$$

This expression equates to zero if

$$\begin{aligned}
&\lim_{\epsilon\to0}\operatorname*{Res}_{w\to u_j}\Big(p(w)\,\Lambda^{(1)}\big(w;\boldsymbol{u}^{(1\dots n-2)},(\boldsymbol{u}^{(n-1)},\tilde{\boldsymbol{u}}-\tfrac{\kappa}{2}-\epsilon);\tilde{\boldsymbol{u}}-\boldsymbol{\delta}\epsilon,\boldsymbol{u}\big)\\
&\qquad\qquad+p(\tilde w)\,\Lambda^{(1)}\big(\tilde w;\boldsymbol{u}^{(1\dots n-2)},(\boldsymbol{u}^{(n-1)},\tilde{\boldsymbol{u}}-\tfrac{\kappa}{2}-\epsilon);\tilde{\boldsymbol{u}},\boldsymbol{u}+\hat{\boldsymbol{\delta}}\epsilon\big)\Big)\\
&=\lim_{\epsilon\to0}\operatorname*{Res}_{w\to u_j}\bigg(p(w)\,\Lambda^{-}\big(w-\tfrac{n-1}{2},\boldsymbol{u}^{(n-1)}\big)\,\Lambda^{-}\big(w-\tfrac{n-1}{2},\tilde{\boldsymbol{w}}-\tfrac{\kappa}{2}-\epsilon\big)\\
&\qquad\qquad\qquad\times\Lambda^{+}\big(w-\tfrac{n}{2},\tilde{\boldsymbol{w}}-\boldsymbol{\delta}\epsilon-\tfrac{n}{2}\big)\,\Lambda^{+}\big(w-\tfrac{n}{2},\boldsymbol{u}-\tfrac{n}{2}\big)\,\frac{\widetilde{\gamma}_n(w)}{2w-n+1+\rho}\\
&\qquad\qquad+p(\tilde w)\,\Lambda^{-}\big(\tilde w-\tfrac{n-1}{2},\boldsymbol{u}^{(n-1)}\big)\Lambda^{-}\big(\tilde w-\tfrac{n-1}{2},\tilde{\boldsymbol{u}}-\tfrac{\kappa}{2}-\epsilon\big)\\
&\qquad\qquad\qquad\times\Lambda^{+}\big(\tilde w-\tfrac{n}{2},\tilde{\boldsymbol{u}}-\tfrac{n}{2}\big)\,\Lambda^{+}\big(\tilde w-\tfrac{n}{2},\boldsymbol{u}+\hat{\boldsymbol{\delta}}\epsilon-\tfrac{n}{2}\big)\,\frac{\widetilde{\gamma}_n(\tilde w)}{2\tilde w-n+1+\rho}\bigg)=0.
\end{aligned}$$

Note that the terms that contain a pole at u_j are $\Lambda^{+}\big(w-\frac{n}{2},\boldsymbol{u}-\frac{n}{2}\big)$ and $\Lambda^{+}\big(\tilde w-\frac{n}{2},\tilde{\boldsymbol{u}}-\frac{n}{2}\big)$. Now evaluate the residue and use identities $\Lambda^{\pm}(\bar v,\boldsymbol{w})=\Lambda^{\mp}(v,\boldsymbol{w})$, $\Lambda^{\pm}(v,\bar{\boldsymbol{w}})=\Lambda^{\pm}(v,\boldsymbol{w})$ and $p(w)=-p(\tilde w)$. Then, upon rewriting u_i's in terms of $u_i^{(n)}$'s, we obtain

$$\begin{aligned}
&\lim_{\epsilon\to0}\Bigg(\Lambda^{-}\big(u_j^{(n)},\boldsymbol{u}^{(n-1)}\big)\Lambda^{-}\big(u_j^{(n)},\boldsymbol{u}^{(n)}+\epsilon\big)\Lambda^{+}\big(u_j^{(n)}-\tfrac12,\boldsymbol{u}^{(n)}+\boldsymbol{\delta}\epsilon+\tfrac12\big)\\
&\qquad\qquad\times\frac{\widetilde{\gamma}_n(u_j^{(n)}+\frac{\kappa}{2})}{2u_j^{(n)}+\rho-1}\prod_{\substack{i=1\\i\neq j}}^{m_n}\frac{u_j^{(n)}-u_i^{(n)}+1}{u_j^{(n)}-u_i^{(n)}}\,\frac{u_j^{(n)}+u_i^{(n)}+\rho}{u_j^{(n)}+u_i^{(n)}+\rho-1}\\
&\qquad-\Lambda^{+}\big(u_j^{(n)},\boldsymbol{u}^{(n-1)}\big)\Lambda^{+}\big(u_j^{(n)},\boldsymbol{u}^{(n)}+\epsilon\big)\Lambda^{-}\big(u_j^{(n)}+\tfrac12,\boldsymbol{u}^{(n)}+\hat{\boldsymbol{\delta}}\epsilon-\tfrac12\big)\\
&\qquad\qquad\times\frac{\widetilde{\gamma}_n(\bar u_j^{(n)}+\frac{\kappa}{2})}{2u_j^{(n)}+\rho+1}\prod_{\substack{i=1\\i\neq j}}^{m_n}\frac{u_j^{(n)}-u_i^{(n)}-1}{u_j^{(n)}-u_i^{(n)}}\,\frac{u_j^{(n)}+u_i^{(n)}+\rho}{u_j^{(n)}+u_i^{(n)}+\rho+1}\Bigg)=0.
\end{aligned}\tag{3.2.93}$$

Observe that

$$\begin{aligned}
\lim_{\epsilon\to0}\Lambda^{-}\big(u_j^{(n)},\boldsymbol{u}^{(n)}+\epsilon\big)\Lambda^{+}\big(u_j^{(n)}-\tfrac12,\boldsymbol{u}^{(n)}+\boldsymbol{\delta}\epsilon+\tfrac12\big)&=\delta_j\,\frac{2u_j^{(n)}+\rho-1}{2u_j^{(n)}+\rho}\,\frac{2u_j^{(n)}+\rho+1}{2u_j^{(n)}+\rho}\\
\times\prod_{\substack{i=1\\i\neq j}}^{m_n}\frac{u_j^{(n)}-u_i^{(n)}-1}{u_j^{(n)}-u_i^{(n)}}\,\frac{u_j^{(n)}+u_i^{(n)}+\rho-1}{u_j^{(n)}+u_i^{(n)}+\rho}&\,\frac{u_j^{(n)}-u_i^{(n)}}{u_j^{(n)}-u_i^{(n)}-1}\,\frac{u_j^{(n)}+u_i^{(n)}+\rho+1}{u_j^{(n)}+u_i^{(n)}+\rho}
\end{aligned}$$

and

$$\lim_{\epsilon\to 0}\Lambda^+\big(u_j^{(n)},\boldsymbol{u}^{(n)}+\epsilon\big)\Lambda^-\big(u_j^{(n)}+\tfrac{1}{2},\boldsymbol{u}^{(n)}+\hat{\boldsymbol{\delta}}\epsilon-\tfrac{1}{2}\big)=\hat{\delta}_j\,\frac{2u_j^{(n)}+\rho-1}{2u_j^{(n)}+\rho}\,\frac{2u_j^{(n)}+\rho+1}{2u_j^{(n)}+\rho}$$
$$\times\prod_{\substack{i=1\\ i\neq j}}^{m_n}\frac{u_j^{(n)}-u_i^{(n)}+1}{u_j^{(n)}-u_i^{(n)}}\,\frac{u_j^{(n)}+u_i^{(n)}+\rho+1}{u_j^{(n)}+u_i^{(n)}+\rho}\,\frac{u_j^{(n)}-u_i^{(n)}}{u_j^{(n)}-u_i^{(n)}+1}\,\frac{u_j^{(n)}+u_i^{(n)}+\rho-1}{u_j^{(n)}+u_i^{(n)}+\rho}.$$

Hence taking the $\epsilon\to 0$ limit in (3.2.93) gives

$$\Lambda^-\big(u_j^{(n)},\boldsymbol{u}^{(n-1)}\big)\,\delta_j\,\frac{\widetilde{\gamma}_n(u_j^{(n)}+\frac{\kappa}{2})}{2u_j^{(n)}+\rho-1}\prod_{\substack{i=1\\ i\neq j}}^{m_n}\frac{u_j^{(n)}-u_i^{(n)}+1}{u_j^{(n)}-u_i^{(n)}}\,\frac{u_j^{(n)}+u_i^{(n)}+\rho}{u_j^{(n)}+u_i^{(n)}+\rho-1}$$
$$-\Lambda^+\big(u_j^{(n)},\boldsymbol{u}^{(n-1)}\big)\,\hat{\delta}_j\,\frac{\widetilde{\gamma}_n(\bar{u}_j^{(n)}+\frac{\kappa}{2})}{2u_j^{(n)}+\rho+1}\prod_{\substack{i=1\\ i\neq j}}^{m_n}\frac{u_j^{(n)}-u_i^{(n)}-1}{u_j^{(n)}-u_i^{(n)}}\,\frac{u_j^{(n)}+u_i^{(n)}+\rho}{u_j^{(n)}+u_i^{(n)}+\rho+1}=0.$$

Recall that $\hat{\delta}_j=-\delta_j/(1-\delta_j)$. We may thus rewrite the equality above as

$$\frac{2u_j^{(n)}+\rho+1}{2u_j^{(n)}+\rho-1}\,\frac{\widetilde{\gamma}_n(u_j^{(n)}+\frac{\kappa}{2})}{\widetilde{\gamma}_n(\frac{\kappa}{2}-u_j^{(n)}-\rho)}$$
$$\times\prod_{\substack{i=1\\ i\neq j}}^{m_n}\frac{(u_j^{(n)}-u_i^{(n)}+1)(u_j^{(n)}+u_i^{(n)}+\rho+1)}{(u_j^{(n)}-u_i^{(n)}-1)(u_j^{(n)}+u_i^{(n)}+\rho-1)}=-\frac{1}{1-\delta_j}\,\frac{\Lambda^+(u_j^{(n)},\boldsymbol{u}^{(n-1)})}{\Lambda^-(u_j^{(n)},\boldsymbol{u}^{(n-1)})}.$$

Substituting the definition of δ_j from (3.2.87) and using $\Lambda^+(u_j^{(n)}-\frac{1}{2},\boldsymbol{u}^{(n-2)})\,\Lambda^-(u_j^{(n)}+\frac{1}{2},\boldsymbol{u}^{(n-2)})=1$ we obtain (3.2.91), as required. □

Example 3.2.40. The orthogonal Bethe vector with a single top-level excitation and $m_1=\ldots=m_{n-1}=0$ is given by

$$\Psi(u^{(n)})=\Bigg(-\frac{2u^{(n)}+\rho-1}{2u^{(n)}+\rho}\big[B(u^{(n)}+\tfrac{\kappa}{2})\big]_{n-1,1}\big[\hat{A}^{(n-1)}(\bar{u}^{(n)}+\tfrac{1}{2})\big]_{11}$$
$$+\frac{1}{1-\delta}\cdot\frac{1}{2u^{(n)}+\rho+1}\big[B(u^{(n)}+\tfrac{\kappa}{2})\big]_{n-1,1}$$
$$\times\left(\big[\hat{A}^{(n-1)}(\bar{u}^{(n)}+\tfrac{1}{2})\big]_{22}-\frac{\big[\hat{A}^{(n-1)}(\bar{u}^{(n)}+\frac{1}{2})\big]_{11}}{2u^{(n)}+\rho}\right)$$
$$-\frac{1}{1-\delta}\cdot\frac{2u^{(n)}+\rho}{2u^{(n)}+\rho+1}\big[B(u^{(n)}+\tfrac{\kappa}{2})\big]_{n,2}$$
$$\times\left(\big[\hat{A}^{(n-1)}(\bar{u}^{(n)}+\tfrac{1}{2})\big]_{22}-\frac{\big[\hat{A}^{(n-1)}(\bar{u}^{(n)}+\frac{1}{2})\big]_{11}}{2u^{(n)}+\rho}\right)\Bigg)\cdot\xi,$$

where $\hat{A}^{(n-1)}(v)$ refers to the level-$(n-1)$ nested version of the A operator of $S(v)$ obtained via (3.1.13). Note that the level-$(n-1)$ excitations contribute only diagonal elements, which do not modify the vacuum vector. Hence the expression above may be simplified by using (3.2.87) and

$$\big[\hat{A}^{(n)}(\bar{u}^{(n)})\big]_{11}\cdot\xi = -\frac{\widetilde{\gamma}_n(\bar{u}^{(n)}+\frac{\kappa}{2})}{2u^{(n)}+\rho}\,\xi, \qquad \big[\hat{A}^{(n-1)}(\bar{u}^{(n)}+\tfrac{1}{2})\big]_{11}\cdot\xi = -\frac{\widetilde{\gamma}_{n-1}(\bar{u}^{(n)}+\frac{\kappa}{2})}{2u^{(n)}+\rho-1}\,\xi, x$$

resulting in

$$\Psi(u^{(n)}) = \frac{\widetilde{\gamma}_{n-1}(\bar{u}^{(n)}+\frac{\kappa}{2})}{2u^{(n)}+\rho-1}\left(\big[S(u^{(n)}+\tfrac{\kappa}{2})\big]_{n-1,n+1} - \big[S(u^{(n)}+\tfrac{\kappa}{2})\big]_{n,n+2}\right)\cdot\xi.$$

3.2.11 Hamiltonian for the fundamental open spin chain

In this section, we discuss the case in which each bulk quantum space is the fundamental representation of $\mathfrak{g}_{2n}$ and each $c_i = -\kappa/2$, i.e., $M \cong (\mathbb{C}^{2n})^{\otimes\ell}$. Additionally, set $\rho = 0$. Let $K(u)$ denote the K-matrix associated to a one-dimensional representation of $X_\rho(\mathfrak{g}_{2n}, \mathfrak{g}_{2n}^{\theta})^{tw}$, as listed in Proposition 3.2.8. Additionally, let $K^*(u)$ denote a solution of the dual reflection (obtained by substituting $u \to \tilde{u}$ and $v \to \tilde{v}$ in the reflection equation, so that $K^*(u) = K(\tilde{u})$). Note that in the above Sections we have taken $K^*(u) = I$. For such an open spin chain, the transfer matrix given in Definition 3.2.24 takes the form

$$\tau(u) = \frac{g(u)}{2u-2\kappa}\,\mathrm{tr}_a\left(K_a^*(u)\left(\prod_{i=1}^{\ell} R_{ai}(u)\right)K_a(u)\left(\prod_{i=\ell}^{1} R_{ai}(u)\right)\right).$$

Prior to extracting a Hamiltonian, we may cancel the poles at $u = 0$ and $u = \kappa$ by multiplying by a certain rational function in u to obtain

$$t(u) = \mathrm{tr}_a\left(\mathbb{K}_a^*(u)\left(\prod_{i=1}^{\ell}\mathbb{R}_{ai}(u)\right)\mathbb{K}_a(u)\left(\prod_{i=\ell}^{1}\mathbb{R}_{ai}(u)\right)\right), \tag{3.2.94}$$

where

$$\mathbb{R}(u) = -\frac{u(\kappa-u)}{\kappa}R(u) = -\frac{u(\kappa-u)}{\kappa} + \frac{\kappa-u}{\kappa}P + \frac{u}{\kappa}Q \in \mathrm{End}(\mathbb{C}^{2n}\otimes\mathbb{C}^{2n})[u],$$

and $\mathbb{K}(u), \mathbb{K}^*(u)$ are normalised such that $\mathbb{K}(0) = \mathbb{K}^*(\kappa) = I$, with $\mathrm{tr}\,\mathbb{K}(\kappa)$ and $\mathrm{tr}\,\mathbb{K}^*(0)$ both non-zero.

Proposition 3.2.41. *The following Hamiltonian commutes with $\tau(u)$:*

$$H^0 = H_L^0 + \sum_{i=1}^{\ell-1} H_{i,i+1}^0 + H_R^0, \tag{3.2.95}$$

where

$$H_L^0 = \frac{\mathrm{tr}_a\left(\mathbb{K}_a^*(0)H_{a1}^0\right)}{\mathrm{tr}\,\mathbb{K}^*(0)}, \qquad H_R^0 = \tfrac{1}{2}\mathbb{K}_\ell'(0), \qquad H_{i,i+1}^0 = P_{i,i+1} \pm \frac{Q_{i,i+1}}{\kappa}.$$

Proof. Observe that $\mathbb{R}(0) = P$, and $\mathbb{K}(0) = I$, so Proposition 4 in [Sk88] allows us to extract a nearest neighbour interaction Hamiltonian for the system. □

The Hamiltonian (3.2.95) is equivalent to the one considered in [GKR05]. The two-site interaction term $H_{i,i+1}$ is equivalent to that given in [Rs85].

An additional Hamiltonian may be extracted from $t(u)$ by looking instead at the point $u = \kappa$. At this point, $\mathbb{R}(\kappa)$ is equal to Q, rather than P. Nevertheless, the following procedure allows a nearest neighbour interaction Hamiltonian to be extracted.

Proposition 3.2.42. *The following Hamiltonian commutes with $\tau(u)$:*

$$H^\kappa = H_L^\kappa + \sum_{i=1}^{\ell-1} H_{i,i+1}^\kappa + H_R^\kappa, \tag{3.2.96}$$

with

$$H_L^\kappa = \tfrac{1}{2}\big(\mathbb{K}_1^{*\prime}(\kappa)\big)^t, \qquad H_R^\kappa = \frac{\operatorname{tr}_a \big(H_{a\ell}^\kappa \mathbb{K}_a^t(\kappa)\big)}{\operatorname{tr}_a \mathbb{K}_a^t(\kappa)}, \qquad H_{i,i+1}^\kappa = P_{i,i+1} \mp \frac{Q_{i,i+1}}{\kappa}.$$

Proof. We begin by differentiating $t(u)$ at $u = \kappa$ to obtain

$$\begin{aligned} t'(\kappa) = \operatorname{tr}_a \Bigg(& \mathbb{K}_a^{*\prime}(\kappa)\, Q_{a1} \cdots Q_{a\ell}\, \mathbb{K}_a(\kappa)\, Q_{a\ell} \cdots Q_{a1} + Q_{a1} \cdots Q_{a\ell}\, \mathbb{K}_a'(\kappa)\, Q_{a\ell} \cdots Q_{a1} \\ & + \sum_{j=1}^{\ell} Q_{a1} \cdots \mathbb{R}_{aj}'(\kappa) \cdots Q_{a\ell}\, \mathbb{K}_a(\kappa)\, Q_{a\ell} \cdots Q_{a1} \\ & + \sum_{j=1}^{\ell} Q_{a1} \cdots Q_{a\ell}\, \mathbb{K}_a(\kappa)\, Q_{a\ell} \cdots \mathbb{R}_{aj}'(\kappa) \cdots Q_{a1} \Bigg). \end{aligned}$$

Repeated applications of $Q_{ai} M_a Q_{ai} = Q_{ai} \operatorname{tr} M$ and $\operatorname{tr}_a Q_{ai} = I$ allow us to reduce this to:

$$\begin{aligned} t'(\kappa) = {} & \operatorname{tr}_a \big(\mathbb{K}_a^{*\prime}(\kappa) Q_{a1}\big) \operatorname{tr}_b \mathbb{K}_b(\kappa) + \operatorname{tr} \mathbb{K}'(\kappa) \\ & + \sum_{j=1}^{\ell-1} \operatorname{tr}_a \big(\mathbb{R}_{aj}'(\kappa) Q_{a,j+1} Q_{aj}\big) \operatorname{tr}_b \mathbb{K}_b(\kappa) + \operatorname{tr}_a \big(\mathbb{R}_{a\ell}'(\kappa) \mathbb{K}_a(\kappa) Q_{a\ell}\big) \\ & + \sum_{j=1}^{\ell-1} \operatorname{tr}_a \big(Q_{aj} Q_{a,j+1} \mathbb{R}_{aj}'(\kappa)\big) \operatorname{tr}_b \mathbb{K}_b(\kappa) + \operatorname{tr}_a \big(Q_{a\ell} \mathbb{K}_a(\kappa) \mathbb{R}_{a\ell}'(\kappa)\big). \end{aligned}$$

Since $\mathbb{R}'(\kappa) = I - P/\kappa + Q/\kappa$, it commutes with Q acting on the same spaces, allowing us to apply the cyclicity of the partial trace. With this, and the identity $Q_{ai} M_a = Q_{ai} M_i^t$, we obtain

$$\begin{aligned} t'(\kappa) = {} & \big(\mathbb{K}_1^{*\prime}(\kappa)\big)^t \operatorname{tr}_a \mathbb{K}_a(\kappa) + \operatorname{tr}_a \mathbb{K}_a'(\kappa) \\ & + 2 \operatorname{tr}_a \mathbb{K}_a(\kappa) \sum_{j=1}^{\ell-1} \mathbb{R}_{j,j+1}'(\kappa)^t P_{j,j+1} + 2 \operatorname{tr}_a \big(\mathbb{K}_a(\kappa) \mathbb{R}_{\ell a}'(\kappa) Q_{\ell a}\big). \end{aligned}$$

From here we divide by $\operatorname{tr}_a \mathbb{K}_a(\kappa)$ and subtract appropriate constants to extract the Hamiltonian. $\square$

Remark 3.2.43. Note that in the case where both conditions on $\mathbb{K}$ and $\mathbb{K}^*$ hold, the Hamiltonian $H^0 + H^\kappa$ has nearest neighbour interaction in the bulk given by $P_{i,i+1}$.

Chapter 4

Nested algebraic Bethe ansatz for q-deformed even orthogonal and symplectic closed spin chains

In this chapter we present the nested algebraic Bethe ansatz for a closed spin chain with underlying algebra given by the $\mathfrak{so}_{2n}$ or $\mathfrak{sp}_{2n}$ quantum loop algebra. The spin chain sites are given by $U_q(\mathfrak{so}_{2n})$ or $U_q(\mathfrak{sp}_{2n})$ representations, with the action extended to the quantum loop algebra via an analogue of the fusion procedure introduced in Chapter 3. The nested algebraic Bethe ansatz is then undertaken using the same nesting procedure as the previous two chapters, the nested system being the q-deformed $\mathfrak{gl}_n$ closed spin chain. As with the system in Chapter 3, care must be taken to distinguish the orthogonal and symplectic cases; we find that the auxiliary spaces at the top level appear as fused (anti-)symmetric representations of $U_q(\mathfrak{gl}_n)$ in the nested system. The eigenvalues and Bethe equations are found, and a closed 'trace' formula for the eigenvectors is given, analogous to the one given in Chapter 2.

4.1 Preliminaries and definitions

4.1.1 Quantised enveloping algebras $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{g}_{2n})$

We must first define the q-deformed equivalents to the classical Lie algebras. Recall that in Chapter 1 we gave the definitions of the classical enveloping algebras $U(\mathfrak{gl}_n)$ and $U(\mathfrak{g}_N)$, in terms of their generators and commutation relations. We then gave an equivalent, matrix form of the defining relations (1.2.2) and (1.2.21). It will be simplest to use this latter style of definition for the quantised enveloping algebras $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{g}_{2n})$.

Let $q \in \mathbb{R}^\times$ with $q \neq \pm 1$. Following [J86a], the algebra $U_q(\mathfrak{gl}_n)$ may be defined as an RTT algebra associated to a constant R-matrix

$$R_q = \sum_{i,j=1}^{n} q^{\delta_{ij}} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i<j} e_{ij} \otimes e_{ji}, \tag{4.1.1}$$

which satisfies the constant Yang-Baxter equation

$$(R_q)_{12}(R_q)_{13}(R_q)_{23} = (R_q)_{23}(R_q)_{13}(R_q)_{12}.$$

That is, we define the generating matrices $L^{\pm} = \sum_{i,j=1}^{N} e_{ij} \otimes \ell_{ij}^{\pm} \in \mathrm{End}(\mathbb{C}^N) \otimes U_q(\mathfrak{gl}_n)$, which are triangular matrices

$$\ell_{ij}^{+} = 0 \quad \text{for} \quad i < j, \qquad \ell_{ij}^{-} = 0 \quad \text{for} \quad i > j.$$

Then the defining relations of $U_q(\mathfrak{gl}_n)$ are

$$\begin{aligned} R_q L_1^{\pm} L_2^{\pm} &= L_2^{\pm} L_1^{\pm} R_q, \\ R_q L_1^{+} L_2^{-} &= L_2^{-} L_1^{+} R_q, \\ \ell_{ii}^{+}\ell_{ii}^{-} = \ell_{ii}^{-}\ell_{ii}^{+} &= 1 \quad \text{for} \quad 1 \le i \le n. \end{aligned} \tag{4.1.2}$$

Following [FRT90] and [GRW20], the algebra $U_q(\mathfrak{g}_{2n})$ may be defined in the same way, although we must first introduce the following notation. We use parameter $\theta = 1$ for the $\mathfrak{so}_{2n}$ case and $\theta = -1$ for the $\mathfrak{sp}_{2n}$ case. Note that in previous chapters this role was taken by the double sign symbols $\pm$, $\mp$; we make this change of notation in order to match the literature on the respective algebras in each chapter. We will also make use of the notation $\theta_{ij} = \theta_i\theta_j$ with $\theta_i = \theta$ if $1 \le i \le n$ and $\theta_i = 1$ if $n < i \le 2n$. Now, defining also $\theta' = \frac{1}{2}(\theta + 1)$—that is, the indicator function for the orthogonal case—we introduce the tuple of integers

$$(\nu_1, \ldots, \nu_{2n}) = (-n + \theta', -n + 1 + \theta', \ldots, -1 + \theta', 1 - \theta', \ldots, n - 1 - \theta', n - \theta'). \tag{4.1.3}$$

This allows us to define the constant R-matrix for $U_q(\mathfrak{g}_{2n})$,

$$R_q = \sum_{i,j=1}^{2n} q^{\delta_{ij} - \delta_{i,2n-j+1}} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i<j} (e_{ij} \otimes e_{ji} - q^{\nu_i - \nu_j} \theta_{ij} e_{ij} \otimes e_{2n-i+1,2n-j+1}). \tag{4.1.4}$$

We also define a matrix transposition ω, which may be thought of as a q-analogue of the matrix transposition t from previous chapters:

$$(e_{ij})^{\omega} = q^{\nu_i - \nu_j} \theta_{ij} e_{2n-j+1,2n-i+1}. \tag{4.1.5}$$

This transpose, however, is not involutive, and we separately define its inverse as $\bar{\omega} : e_{ij} \mapsto q^{\nu_j - \nu_i} \theta_{ij} e_{2n-j+1,2n-i+1}$. The algebra $U_q(\mathfrak{g}_{2n})$ is then the associative algebra generated by relations

(4.1.2) with R-matrix (4.1.4), along with a cross-unitarity relation

$$L^{\pm}(L^{\pm})^{\bar{\omega}} = (L^{\pm})^{\bar{\omega}} L^{\pm} = I. \tag{4.1.6}$$

4.1.2 Quantum loop algebras $U_q(\mathfrak{L}\mathfrak{g}_{2n})$ and $U_q(\mathfrak{L}\mathfrak{gl}_n)$

We now introduce the matrix operators required to define the quantum loop algebras $U_q(\mathfrak{L}\mathfrak{g}_{2n})$ and $U_q(\mathfrak{L}\mathfrak{gl}_n)$. The R matrix for the former is given by

$$R(u,v) := R_q + \frac{q-q^{-1}}{v/u-1} P - \frac{q-q^{-1}}{q^{2\kappa}v/u-1} Q_q, \tag{4.1.7}$$

where R_q is given by (4.1.4), P is the permutation operator $P = \sum_{i,j} e_{ij} \otimes e_{ji}$ and $Q_q = P^{\bar{\omega}_2}$. The matrix $R(u,v)$, obtained by Jimbo in [J86b], is a solution of the quantum Yang-Baxter equation on $(\mathbb{C}^{2n})^{\otimes 3}$ with spectral parameters,

$$R_{12}(u,v)\, R_{13}(u,w)\, R_{23}(v,w) = R_{23}(v,w)\, R_{13}(u,w)\, R_{12}(u,v). \tag{4.1.8}$$

In this chapter, the $U_q(\mathfrak{L}\mathfrak{gl}_n)$ model will appear as the nested system after the first level of nesting, and so we introduce the R-matrix and relevant operators in this context. The reduced $U_q(\mathfrak{L}\mathfrak{gl}_n)$ R-matrix on $\mathbb{C}^{n-k+1} \otimes \mathbb{C}^{n-l+1}$, where $k \geq l$ in all cases we are interested in, is given by

$$R^{(k,l)}(u,v) := R_q^{(k,l)} + \frac{q-q^{-1}}{v/u-1} P^{(k,l)}, \tag{4.1.9}$$

where the bracketed superscripts refer to the nesting level on each of the two tensor spaces. The matrix operators here are reduced versions of those introduced above. Indeed, let $e_{ij}^{(k)}$ denote the $(n-k+1) \times (n-k+1)$ elementary matrices. Then

$$\begin{aligned} R_q^{(k,l)} &:= \sum_{i=1}^{n-k+1} \sum_{j=1}^{n-l+1} q^{\delta_{ij}} e_{ii}^{(k)} \otimes e_{jj}^{(l)} + (q-q^{-1}) \sum_{i,j=1}^{n-k+1} \delta_{i<j}\, e_{ij}^{(k)} \otimes e_{j'i'}^{(l)}, \\ P^{(k,l)} &:= \sum_{i,j=1}^{n-k+1} e_{ij}^{(k)} \otimes e_{j'i'}^{(l)}, \qquad Q_q^{(k,l)} := \sum_{i,j=1}^{n-k+1} q^{i-j} e_{ij}^{(k)} \otimes e_{\bar{\imath}\bar{\jmath}}^{(l)}, \end{aligned} \tag{4.1.10}$$

where $i' = i+k-l$ and $j' = j+k-l$, and $\bar{\imath} = (n-l+1)-i+1$, $\bar{\jmath} = (n-l+1)-j+1$. Note here that the q factor for $Q_q^{(k,l)}$ will have a different form to that of Q_q. We also introduce an equivalent of the transpose ω on the reduced spaces by

$$(e_{ij}^{(l)})^{\omega} = q^{i-j} e_{\bar{\jmath}\bar{\imath}}^{(l)} \qquad \text{and} \qquad (e_{ij}^{(l)})^{\bar{\omega}} = q^{j-i} e_{\bar{\jmath}\bar{\imath}}^{(l)}.$$

with $\bar{\imath} = (n-l+1)-i+1$, $\bar{\jmath} = (n-l+1)-j+1$. Despite the difference with (4.1.5), we have used the same notation ω and $\bar{\omega}$; it will be clear which one is meant from the matrix it is acting on.

These matrix operators satisfy

$$(R_q^{(k,l)})^{-1} = R_{q^{-1}}^{(k,l)}, \qquad P^{(k,l)} Q_q^{(k,l)} P^{(k,l)} = Q_{q^{-1}}^{(k,l)}. \tag{4.1.11}$$

Here the subscript q^{-1} means that all instances of q are replaced with q^{-1} in the definition of the operator; such notation will be used throughout this chapter.

Recall that $\mathbb{C}^{2n} \cong \mathbb{C}^2 \otimes \mathbb{C}^n$. Let x_{ij} with $1 \le i,j \le 2$ denote the matrix units of $\mathrm{End}(\mathbb{C}^2)$. Then, for any $1 \le i,j \le n$, we may write

$$e_{ij} = x_{11} \otimes e_{ij}^{(1)}, \quad e_{n+i,j} = x_{21} \otimes e_{ij}^{(1)}, \quad e_{i,n+j} = x_{12} \otimes e_{ij}^{(1)}, \quad e_{n+i,n+j} = x_{22} \otimes e_{ij}^{(1)}. \tag{4.1.12}$$

Viewing the matrix $R(u,v)$ as an element in $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ we recover the six-vertex block structure,

$$R(u,v) = \begin{pmatrix} R^{(1,1)}(u,v) & & & \\ & K^{(1,1)}(u,q^{2\kappa}v) & U^{(1,1)}(u,v) & \\ & \bar{U}^{(1,1)}(v,u) & K^{(1,1)}(u,q^{2\kappa}v) & \\ & & & R^{(1,1)}(u,v) \end{pmatrix}, \tag{4.1.13}$$

where the operators inside the matrix above are each acting on $\mathbb{C}^n \otimes \mathbb{C}^n$ and may be written in terms of those in (4.1.10) as

$$\begin{gathered} U^{(k,k)}(u,v) := -\frac{q-q^{-1}}{u/v-1} P^{(k,k)} + \frac{\theta(q-q^{-1})}{q^{-\kappa'}u/v - q^{\kappa'}} Q_q^{(k,k)}, \\ \bar{U}^{(k,k)}(v,u) := P^{(k,k)} U_{q^{-1}}^{(k,k)}(v,u) P^{(k,k)}, \qquad K^{(k,l)}(u,v) := (R_{q^{-1}}^{(k,l)}(u,v))^{\bar{\omega}_2}, \end{gathered} \tag{4.1.14}$$

where $\kappa' = \kappa - k + 1$. We also note two more important identities,

$$\begin{aligned} \bar{U}^{(k,k)}(v,u)\,(K^{(k,k)}(u,q^{2\kappa}v))^{-1} &= \frac{q-q^{-1}}{v/u-1} P^{(k,k)} (K^{(k,k)}(u,q^{2\kappa}u))^{-1} \\ &= \frac{1}{v-u} \operatorname*{Res}_{w\to u} R^{(k,k)}(u,w)(K^{(k,k)}(u,q^{2\kappa}w))^{-1}, \end{aligned} \tag{4.1.15}$$

$$\begin{aligned} (K^{(k,k)}(v,q^{2\kappa}u))^{-1} U^{(k,k)}(v,u) &= \frac{q^{-1}-q}{v/u-1} (K^{(k,k)}(u,q^{2\kappa}u))^{-1} P^{(k,k)} \\ &= \frac{1}{v-u} \operatorname*{Res}_{w\to u} (K^{(k,k)}(w,q^{2\kappa}u))^{-1} R^{(k,k)}(w,u), \end{aligned} \tag{4.1.16}$$

that will play a key part in finding the unwanted terms of the algebraic Bethe ansatz. Lastly, introduce elements

$$\begin{gathered} \mathcal{E}_{ij}^{+(l)} := \delta_{ij} q^{-e_{ii}^{(l)}} - (q-q^{-1})\delta_{i<j} e_{ij}^{(l)}, \qquad \mathcal{E}_{ji}^{-(l)} := \delta_{ij} q^{e_{ii}^{(l)}} + (q-q^{-1})\delta_{i<j} e_{ji}^{(l)}, \\ \mathcal{E}_{ij}^{(l)}(u) := \frac{1}{1-u}\mathcal{E}_{ij}^{+(l)} + \frac{1}{1-u^{-1}}\mathcal{E}_{ij}^{-(l)}, \end{gathered} \tag{4.1.17}$$

where we have used the notation $q^{e^{(l)}_{ii}} = \sum_{j=1}^{n-l+1} q^{\delta_{ij}} e^{(l)}_{jj}$. Then we may write

$$R^{(k,l)}(u,v) = \sum_{i,j=1}^{n-k+1} e^{(k)}_{ij} \otimes \mathcal{E}^{(l)}_{j'i'}(v/u), \qquad K^{(k,l)}(u,v) = \sum_{i,j=1}^{n-k+1} e^{(k)}_{ij} \otimes (\mathcal{E}^{(l)}_{q^{-1},j'i'}(v/u))^{\bar{\omega}}, \tag{4.1.18}$$

where $i' = i + k - l$ and $j' = j + k - l$.

We are now ready to define the quantum loop algebras $U_q(\mathfrak{L}\mathfrak{g}_{2n})$ and $U_q(\mathfrak{L}\mathfrak{gl}_n)$. The two algebras may again be defined by RTT relations, with the R-matrices defined above. As such, we combine the two definitions, writing $N = 2n$ or $N = n$ respectively. We then introduce elements $\ell^{\pm}_{ij}[r]$ with $1 \leq i,j \leq N$ and $r \geq 0$, combine them into formal series $\ell^{\pm}_{ij}(u) = \sum_{r\geq 0} \ell^{\pm}_{ij}[r]\, u^{\pm r}$, and collect into generating matrices

$$L^{\pm}(u) := \sum_{1\leq i,j\leq N} e_{ij} \otimes \ell^{\pm}_{ij}(u). \tag{4.1.19}$$

We will say that elements $\ell^{\pm}_{ij}[r]$ have degree r.

Definition 4.1.1. *The quantum loop algebra $U_q(\mathfrak{L}\mathfrak{g}_{2n})$ (resp. $U_q(\mathfrak{L}\mathfrak{gl}_n)$) is the unital associative algebra with generators $\ell^{\pm}_{ij}[r]$ with $1 \leq i,j \leq N$ and $r \geq 0$, subject to the following relations:*

$$\ell^{-}_{ii}[0]\,\ell^{+}_{ii}[0] = 1 \;\text{ for all } i \text{ and }\; \ell^{-}_{ij}[0] = \ell^{+}_{ji}[0] = 0 \;\text{ for } i<j \text{ and} \tag{4.1.20}$$

$$R_{12}(u,v)\, L^{\pm}_1(u)\, L^{\pm}_2(v) = L^{\pm}_2(v)\, L^{\pm}_1(u)\, R_{12}(u,v), \tag{4.1.21}$$

$$R_{12}(u,v)\, L^{+}_1(u)\, L^{-}_2(v) = L^{-}_2(v)\, L^{+}_1(u)\, R_{12}(u,v). \tag{4.1.22}$$

where $N = 2n$ and $R_{12}(u,v)$ is given by (4.1.7) (resp. $N = n$ and $R_{12}(u,v)$ is given by (4.1.9) with $k = l = 1$).

The following subalgebras of $U_q(\mathfrak{L}\mathfrak{g}_{2n})$ will be relevant to the present chapter:

- Looking at the degree 0 elements $\ell^{\pm}_{ij}[0]$ with $1 \leq i,j \leq 2n$, they satisfy the relations (4.1.2), but not the cross-unitarity relation (4.1.6). Thus, they form a subalgebra isomorphic to the quantisation of the direct sum of $\mathfrak{g}_{2n}$ and a one dimensional Lie algebra.

- The subalgebra generated by $\ell^{\pm}_{ij}[r]$ with $1 \leq i,j \leq n$ and $r \geq 0$ is isomorphic to $U_q(\mathfrak{L}\mathfrak{gl}_n)$. This can be seen as a consequence of the decomposition (4.1.13).

- The subalgebra generated by $\ell^{\pm}_{ij}[0]$ with $1 \leq i,j \leq n$ is isomorphic to $U_q(\mathfrak{gl}_n)$.

We now cast the generating matrices $L^{\pm}(u)$ of $U_q(\mathfrak{L}\mathfrak{g}_{2n})$ into $n \times n$ block matrices and find the relations between them. Indeed, we write

$$L^{\pm}(u) = \begin{pmatrix} A^{\pm}(u) & B^{\pm}(u) \\ C^{\pm}(u) & D^{\pm}(u) \end{pmatrix}. \tag{4.1.23}$$

Then, viewing $L_1^{\pm}(u)$ and $L_2^{\pm}(u)$ as elements in $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ with entries in $\mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n) \otimes U_q(\mathfrak{Lg}_{2n})[[u^{\pm 1}]]$, we have

$$L_1^{\pm}(u) = \begin{pmatrix} A_1^{\pm}(u) & & B_1^{\pm}(u) & \\ & A_1^{\pm}(u) & & B_1^{\pm}(u) \\ C_1^{\pm}(u) & & D_1^{\pm}(u) & \\ & C_1^{\pm}(u) & & D_1^{\pm}(u) \end{pmatrix}, \qquad L_2^{\pm}(u) = \begin{pmatrix} A_2^{\pm}(u) & B_2^{\pm}(u) & & \\ C_2^{\pm}(u) & D_2^{\pm}(u) & & \\ & & A_2^{\pm}(u) & B_2^{\pm}(u) \\ & & C_2^{\pm}(u) & D_2^{\pm}(u) \end{pmatrix}.$$

This allows us to write the defining relations of $U_q(\mathfrak{Lg}_{2n})$ in terms of the matrix operators $A^{\pm}(u)$, $B^{\pm}(u)$, $C^{\pm}(u)$ and $D^{\pm}(u)$. The relations that we will need are:

$$A_2^{\pm}(v)\, B_1^{\pm}(u)\, K_{12}^{(1,1)}(u, q^{2\kappa}v) = R_{12}^{(1,1)}(u,v)\, B_1^{\pm}(u) A_2^{\pm}(v) - B_2^{\pm}(v)\, A_1^{\pm}(u)\, \bar{U}_{12}^{(1,1)}(v,u), \tag{4.1.24}$$

$$K_{12}^{(1,1)}(v, q^{2\kappa}u)\, D_1^{\pm}(v)\, B_2^{\pm}(u) = B_2^{\pm}(u)\, D_1^{\pm}(v)\, R_{12}^{(1,1)}(v,u) - U_{12}^{(1,1)}(v,u)\, B_1^{\pm}(v)\, D_2^{\pm}(u), \tag{4.1.25}$$

$$K_{12}^{(1,1)}(u, q^{2\kappa}v)\, C_1^{\pm}(u)\, A_2^{\pm}(v) = A_2^{\pm}(v)\, C_1^{\pm}(u)\, R_{12}^{(1,1)}(u,v) - U_{12}^{(1,1)}(u,v)\, A_1^{\pm}(u)\, C_2^{\pm}(v), \tag{4.1.26}$$

$$C_2^{\pm}(v)\, D_2^{\pm}(u)\, K_{12}^{(1,1)}(u, q^{2\kappa}v) = R_{12}^{(1,1)}(u,v)\, D_1^{\pm}(u)\, C_2^{\pm}(v) - D_2^{\pm}(v)\, C_1^{\pm}(u)\, \bar{U}_{12}^{(1,1)}(v,u), \tag{4.1.27}$$

$$\begin{aligned} K_{12}^{(1,1)}(u, q^{2\kappa}v)\, D_1^{\pm}(u)\, A_2^{\pm}(v) &- A_2^{\pm}(v)\, D_1^{\pm}(u)\, K_{12}^{(1,1)}(u, q^{2\kappa}v) \\ &= B_2^{\pm}(v)\, C_1^{\pm}(u)\, \bar{U}_{12}^{(1,1)}(v,u) - U_{12}^{(1,1)}(u,v)\, B_1^{\pm}(u)\, C_2^{\pm}(v), \end{aligned} \tag{4.1.28}$$

and their mixed counterparts obtained in an obvious way, cf. (4.1.21) vs. (4.1.22). Operators $A^{\pm}(u)$, $B^{\pm}(u)$ and $D^{\pm}(u)$ satisfy relations analogous to (4.1.21) and (4.1.22) only with $R^{(1,1)}(u,v)$, e.g.,

$$R_{12}^{(1,1)}(u,v)\, B_1^{\pm}(u)\, B_2^{\pm}(v) = B_2^{\pm}(v)\, B_1^{\pm}(u)\, R_{12}^{(1,1)}(u,v). \tag{4.1.29}$$

We now focus on the subalgebra $U_q(\mathfrak{gl}_n) \subset U_q(\mathfrak{g}_{2n})$ generated by coefficients of the matrix entries of $A^{\pm}(u)$. Define a k-reduced matrix $A^{\pm(k)}(u) := \sum_{i,j=k}^{n} e_{i-k+1,j-k+1}^{(k)} \otimes [A^{\pm}(u)]_{ij}$ and set

$$a^{\pm(k)}(v) := [A^{\pm(k)}(v)]_{11}, \qquad B^{\pm(k+1)}(u) := \sum_{j=1}^{n-k} (e_j^{(k+1)})^* \otimes [A^{\pm(k)}(u)]_{1,1+j}. \tag{4.1.30}$$

We also define a suitably normalised check $\check{R}$-matrix

$$\check{R}_{12}^{(k,l)}(u,v) := \frac{v-u}{qv - q^{-1}u}\, P_{12}^{(k,l)} R_{12}^{(k,l)}(u,v).$$

The defining relations of $U_q(\mathfrak{Lgl}_n)$ then yield

$$\mathpzc{a}^{\pm(k)}(v)\,B_1^{\pm(k+1)}(u) = \frac{qv - q^{-1}u}{v-u}\,B_1^{\pm(k+1)}(u)\,\mathpzc{a}^{\pm(k)}(v) + \frac{q-q^{-1}}{u/v-1}\,B_1^{\pm(k+1)}(v)\,\mathpzc{a}^{\pm(k)}(u), \tag{4.1.31}$$

$$A_1^{\pm(k)}(v)\,B_2^{\pm(k)}(u) = B_2^{\pm(k)}(u)\,A_1^{\pm(k)}(v)\,R_{12}^{(k,k)}(v,u) - \frac{q-q^{-1}}{u/v-1}\,B_2^{\pm(k)}(v)\,A_1^{\pm(k)}(u)\,P_{12}^{(k,k)}, \tag{4.1.32}$$

$$B_1^{\pm(k)}(u)\,B_2^{\pm(k)}(v) = B_1^{\pm(k)}(v)\,B_2^{\pm(k)}(u)\,\check{R}_{12}^{(k,k)}(u,v), \tag{4.1.33}$$

$$R_{12}^{(k,k)}(u,v)\,A_1^{\pm(k)}(u)\,A_2^{\pm(k)}(v) = A_2^{\pm(k)}(v)\,A_1^{\pm(k)}(u)\,R_{12}^{(k,k)}(u,v), \tag{4.1.34}$$

plus the mixed relations. Note that $R_{12}^{(n,n)}(u,v)$ acts as a constant on $\mathbb{C}\otimes\mathbb{C}\cong\mathbb{C}$ multiplying by $\frac{qv-q^{-1}u}{v-u}$, while $\check{R}_{12}^{(n,n)}(u,v)$ multiplies by 1.

4.1.3 Representations

Just as in Chapter 3, we will need to construct representations of this quantum group using the fusion procedure, and this construction will proceed in an analogous way. For this we follow [IMO12, IMO14].

Each site in the spin chain is built using the fusion procedure, which allows us to define an action of $U_q(\mathfrak{Lg}_{2n})$, on symmetric (respectively skewsymmetric) modules of $U_q(\mathfrak{so}_{2n})$ ($U_q(\mathfrak{sp}_{2n})$). These modules can be obtained by projecting to the (skew)symmetric subspace of a tensor product of s copies of the vector representation $\mathbb{C}^{2n}$, and they define irreducible representations of their respective algebras. Indeed, the symmetric case has lowest weight vector $\eta_s = (e_1)^{\otimes s}$ with weight $\lambda = (q^s, 1, \ldots, 1, q^{-s})$. In the skewsymmetric case, we must restrict to $1 \le s \le n$, and the lowest weight vector is $\eta_s = \sum_{\sigma\in\mathfrak{S}_s} \operatorname{sign}(\sigma)\, q^{l(\sigma)} \cdot e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(s)}$; here $l(\sigma)$ denotes the length of a reduced expression of $\sigma \in \mathfrak{S}_s$, an element in the symmetric group on s letters (full details will be given in [GRW20]). The weight of this vector is given by $\lambda = \big(q, \ldots, q, 1, \ldots, 1, q^{-1}, \ldots, q^{-1}\big)$, where the number of q's and the number of q^{-1}'s is s.

The $U_q(\mathfrak{Lg}_{2n})$-action on either of these modules derives from the action on the tensor product space $(\mathbb{C}^{2n})^{\otimes s}$, which is given by a product of R-matrices. By carefully choosing the relative shift parameters between these R-matrices, we can obtain an action which leaves invariant the (skew)symmetric subspace of the tensor product space. Indeed, letting $c \in \mathbb{C}^\times$ denote an overall shift for this particular site, the map is given by

$$L_a^{\pm}(u) \mapsto \prod_{j=1}^{s} R_{aj}(u, q^{2\theta(j-1)}c).$$

The projector to the (skew)symmetric subspace may also be written as a product of R-matrices,

$$\Pi_s^{\theta} := \frac{\theta}{[s]_q!} \prod_{j=s}^{2} \Big(R_{12}(q^{-2\theta}, 1) P_{12} \cdots R_{j-1,j}(q^{-(j-1)\theta}, 1) P_{j-1,j} \Big).$$

Thus, the space $\Pi_s^\theta(\mathbb{C}^{2n})^{\otimes s}$ may be viewed as a $U_q(\mathfrak{L}\mathfrak{g}_{2n})$-module, which we will denote by $L(\lambda)_c$.

Now, fix $\ell \in \mathbb{N}$, the length of the spin chain. For each $1 \le i \le \ell$ we choose $c_i \in \mathbb{C}^\times$ and $s_i \in \mathbb{N}$, and construct the module $L(\lambda^{(i)})_{c_i}$. As a $U_q(\mathfrak{L}\mathfrak{g}_{2n})$-module, we find that this is a lowest weight module with lowest weight vector η_{s_i} given above, and weights now given by

$$\lambda_j^{(i)}(u, c_i) := \begin{cases} \dfrac{q^{s_i} c_i - q^{-s_i} u}{c_i - u} & \text{if} \quad j = 1, \\ 1 & \text{if} \quad 1 < j < 2n, \\ \dfrac{q^{-1} c_i - q^{-2\kappa+1} u}{q^{s_i - 1} c_i - q^{-2\kappa - s_i + 1} u} & \text{if} \quad j = 2n \end{cases} \tag{4.1.35}$$

in the symmetric case, i.e. when $\mathfrak{g}_{2n} = \mathfrak{so}_{2n}$, and by

$$\lambda_j^{(i)}(u, c_i) := \begin{cases} \dfrac{q\, c_i - q^{-1} u}{c_i - u} & \text{if} \quad 1 \le j \le s_i, \\ 1 & \text{if} \quad s_i < j < 2n - s_i + 1, \\ \dfrac{q^{-s_i} c_i - q^{-2\kappa+s_i} u}{q^{-s_i+1} c_i - q^{-2\kappa + s_i - 1} u} & \text{if} \quad 2n - s_i + 1 \le j \le 2n \end{cases} \tag{4.1.36}$$

in the skewsymmetric case. Here weights $\lambda^{(i)}(u)$ should be expanded as a series in positive (resp. negative) powers of u for $\ell_{jj}^+(u)$ (resp. for $\ell_{jj}^-(u)$).

The full spin chain, on which we will study the transfer matrix spectral problem, is then given by

$$L := L(\lambda^{(1)})_{c_1} \otimes \cdots \otimes L(\lambda^{(\ell)})_{c_\ell}. \tag{4.1.37}$$

The generating matrix $L_a^\pm(u)$ acts on the space L in terms of a product of R-matrices (4.1.7),

$$T_a(u; \boldsymbol{c}) := \prod_{i=1}^{\ell} \prod_{j=1}^{s_i} R_{a i_j}(u, q^{2\theta(j-1)} c_i) \tag{4.1.38}$$

where i_j enumerate individual tensorands $\mathbb{C}^{2n}$ of $L(\lambda^{(i)})_{c_i}$. We will often omit the dependence on $\boldsymbol{c}$ to ease the notation and write $T_a(u)$, its matrix elements will be denoted as $t_{ij}(u)$. The module L is a $U_q(\mathfrak{L}\mathfrak{g}_{2n})$ lowest weight module, with lowest weight vector $\eta = \eta_{s_1} \otimes \cdots \otimes \eta_{s_\ell}$ and lowest weight equal to the product of lowest weights of the individual modules, that is,

$$\ell_{ij}^\pm(u)\eta = 0 \;\text{ for }\; i > j \;\text{ and }\; \ell_{jj}^\pm(u)\eta = \prod_{i=1}^{\ell} \lambda_j^{(i)}(u, c_i)\eta \;\text{ for all } j. \tag{4.1.39}$$

Finally, of particular interest for the nested algebraic Bethe ansatz, we note that the subspace

$$(L(\lambda^{(i)})_{c_i})^0 := \{\xi \in L(\lambda^{(i)})_{c_i} : t_{n+i,j}[0]\,\xi = 0 \text{ for } 1 \le i, j \le n\}$$

is an irreducible $U_q(\mathfrak{gl}_n)$-module of lowest weight $(\lambda_1^{(i)}, \ldots, \lambda_n^{(i)})$. In particular, we have that

$$L^0 := \{\xi \in L : t_{n+i,j}(u)\xi = 0 \text{ for } 1 \le i, j \le n\} = L(\lambda^{(1)})^0_{c_1} \otimes \cdots \otimes L(\lambda^{(\ell)})^0_{c_\ell}.$$

The $A_a^{\pm}(u)$ and $D_a^{\pm}(u)$ operators act on the subspace L^0 in terms of a product of the "reduced" R- and K-matrices defined in (4.1.9) and (4.1.14),

$$A_a^{(1)}(u) := \prod_{i=1}^{\ell} \prod_{j=1}^{s_i} R_{ai_j}^{(1,1)}(u, q^{2\theta(j-1)}c_i), \tag{4.1.40}$$

$$D_a^{(1)}(u) := \prod_{i=1}^{\ell} \prod_{j=1}^{s_i} K_{ai_j}^{(1,1)}(u, q^{2\kappa+2\theta(j-1)}c_i). \tag{4.1.41}$$

4.2 Nested algebraic Bethe ansatz

4.2.1 Quantum spaces and monodromy matrices

Choose $m_0, m_1, \ldots, m_{n-1} \in \mathbb{Z}_{\ge 0}$, which will denote the number of excitations at each level of nesting. For each m_k assign an m_k-tuple $\boldsymbol{u}^{(k)} := (u_1^{(k)}, \ldots, u_{m_k}^{(k)})$ of complex parameters and an m_k-tuple of labels $\boldsymbol{a}^k := (a_1^k, \ldots, a_{m_k}^k)$, which will label the auxiliary spaces. For the top level m_0 the creation operator will live in two auxiliary spaces, and so we additionally assign a tuple $\tilde{\boldsymbol{a}}^0 := (\tilde{a}_1^0, \ldots, \tilde{a}_{m_0}^0)$. As in Chapter 3, we will often use the following shorthand notation:

$$\boldsymbol{u}^{(0\ldots k)} := (\boldsymbol{u}^{(0)}, \ldots, \boldsymbol{u}^{(k)}), \qquad \boldsymbol{a}^{\tilde{0},0\ldots k} := (\tilde{\boldsymbol{a}}^0, \boldsymbol{a}^0, \ldots, \boldsymbol{a}^k). \tag{4.2.1}$$

Let $V_{a_i^{k-1}}^{(k)}$ denote a copy of $\mathbb{C}^{n-k+1}$ and let $W_{\boldsymbol{a}^{k-1}}^{(k)}$ be given by

$$W_{\boldsymbol{a}^{k-1}}^{(k)} := V_{a_1^{k-1}}^{(k)} \otimes \cdots \otimes V_{a_{m_{k-1}}^{k-1}}^{(k)}.$$

Let L be a lowest weight $U_q(\mathfrak{Lg}_{2n})$-module defined in (4.1.37). This will play the role of the original spin chain, and we will refer to it as the level-0 quantum space. Then, starting from $L = L^{(0)}$, we recursively define the nested quantum spaces in the following way. We define the level-1 quantum space by

$$L^{(1)} := (L^{(0)})^0 \otimes W_{\tilde{\boldsymbol{a}}^0}^{(1)} \otimes W_{\boldsymbol{a}^0}^{(1)}, \tag{4.2.2}$$

and each subsequent level-k quantum space, for $2 \le k \le n$, by

$$L^{(k)} := (L^{(k-1)})^0 \otimes W_{\boldsymbol{a}^{k-1}}^{(k)}, \tag{4.2.3}$$

where

$$(L^{(k-1)})^0 := \{\xi \in L^{(k-1)} : t_{ij}(u)\xi = 0 \text{ for } i > j \text{ and } j < k-1\}.$$

As the name quantum space suggests, we will regard these quantum spaces as spin chains in their own right. As such, we recursively define a monodromy matrix for each nested spin chain.

Definition 4.2.1. *We will say that $T_a(v)$ is a level-0 monodromy matrix. We define level-1 monodromy matrices, acting on the space $L^{(1)}$, by*

$$A^{(1)}_{a\boldsymbol{a}^{\tilde{0},0}}(v;\boldsymbol{u}^{(0)}) := \left(\prod_{i=m_0}^{1} K^{(1,1)}_{a\tilde{a}^0_i}(v, q^{2\theta}u^{(0)}_i)\right)\left(\prod_{i=1}^{m_0} K^{(1,1)}_{aa^0_i}(v, u^{(0)}_i)\right) A^{(1)}_a(v), \tag{4.2.4}$$

$$D^{(1)}_{a\boldsymbol{a}^{\tilde{0},0}}(v;\boldsymbol{u}^{(0)}) := D^{(1)}_a(v)\left(\prod_{i=m_0}^{1} R^{(1,1)}_{a\tilde{a}^0_i}(v, u^{(0)}_i)\right)\left(\prod_{i=1}^{m_0} R^{(1,1)}_{aa^0_i}(v, q^{-2\theta}u^{(0)}_i)\right). \tag{4.2.5}$$

For each $2 \le k \le n$ we recursively define level-k monodromy matrices, acting on the spaces $L^{(k)}$, by

$$A^{(k)}_{a\boldsymbol{a}^{\tilde{0},0\ldots k-1}}(v;\boldsymbol{u}^{(0\ldots k-1)}) := A^{(k)}_{a\boldsymbol{a}^{\tilde{0},0\ldots k-2}}(v;\boldsymbol{u}^{(0\ldots k-2)})\left(\prod_{i=1}^{m_{k-1}} R^{(k,k)}_{aa^{k-1}_i}(v, u^{(k-1)}_i)\right), \tag{4.2.6}$$

$$D^{(k)}_{a\boldsymbol{a}^{\tilde{0},0\ldots k-1}}(v;\boldsymbol{u}^{(0\ldots k-1)}) := \left(\prod_{i=1}^{m_{k-1}} R^{(k,k)}_{aa^{k-1}_i}(v, q^{2\kappa'}u^{(k-1)}_i)\right) D^{(k)}_{a\boldsymbol{a}^{\tilde{0},0\ldots k-2}}(v;\boldsymbol{u}^{(0\ldots k-2)}), \tag{4.2.7}$$

where $A^{(k)}_{a\boldsymbol{a}^{\tilde{0},0\ldots k-2}}$ and $D^{(k)}_{a\boldsymbol{a}^{\tilde{0},0\ldots k-2}}$ denote the 1-reduced operators of $A^{(k-1)}_{a\boldsymbol{a}^{\tilde{0},0\ldots k-2}}$ and $D^{(k-1)}_{a\boldsymbol{a}^{\tilde{0},0\ldots k-2}}$, respectively.

Operators $A^{(k)}_{a\boldsymbol{a}^{\tilde{0},0\ldots k-1}}$ and $D^{(k)}_{a\boldsymbol{a}^{\tilde{0},0\ldots k-1}}$ are matrices with entries in $\mathrm{End}(L^{(k)})$. Thus, to ease the notation, we will write them as $A^{(k)}_a$ and $D^{(k)}_a$. We will use a similar notation throughout this chapter.

Lemma 4.2.2. *For $1 \le k \le n-1$ let $\equiv$ denote equality of operators in the space $L^{(k)}$. Then*

$$\begin{aligned}
&R^{(k,k)}_{ab}(v,w)\, A^{(k)}_a(v;\boldsymbol{u}^{(0\ldots k-1)})\, A^{(k)}_b(w;\boldsymbol{u}^{(0\ldots k-1)}) \\
&\qquad \equiv A^{(k)}_b(w;\boldsymbol{u}^{(0\ldots k-1)})\, A^{(k)}_a(v;\boldsymbol{u}^{(0\ldots k-1)})\, R^{(k,k)}_{ab}(v,w), \\
&R^{(k,k)}_{ab}(v,w)\, D^{(k)}_a(v;\boldsymbol{u}^{(0\ldots k-1)})\, D^{(k)}_b(w;\boldsymbol{u}^{(0\ldots k-1)}) \\
&\qquad \equiv D^{(k)}_b(w;\boldsymbol{u}^{(0\ldots k-1)})\, D^{(k)}_a(v;\boldsymbol{u}^{(0\ldots k-1)})\, R^{(k,k)}_{ab}(v,w), \\
&D^{(k)}_a(v;\boldsymbol{u}^{(0\ldots k-1)})\, R^{(k,k)}_{q^{-1},ab}(v,q^{2\kappa'}w)\, A^{(k)}_b(w;\boldsymbol{u}^{(0\ldots k-1)}) \\
&\qquad \equiv A^{(k)}_b(w;\boldsymbol{u}^{(0\ldots k-1)})\, R^{(k,k)}_{q^{-1},ab}(v,q^{2\kappa'}w)\, D^{(k)}_a(v;\boldsymbol{u}^{(0\ldots k-1)})
\end{aligned} \tag{4.2.8}$$

where $\kappa' = \kappa - k + 1$.

Proof. The first two identities follow from the Yang-Baxter equation and the defining relations of A and D operators. For the third identity we additionally need to use the property $C^{\pm}_a(v) \equiv 0$. $\square$

Observe that K-matrices in (4.2.4) and R-matrices in (4.2.5) are "at the fusion point". More precisely, introduce (anti-)symmetric projector $\Pi^{\pm} := \pm\frac{1}{q+q^{-1}} R^{(1,1)}(q^{\mp 2},1)P^{(1,1)}$ and set $V^{\pm} :=$

$\Pi^{\pm}\,\mathbb{C}^n \otimes \mathbb{C}^n$. The subspace $V^{\pm}$ is an irreducible $U_q(\mathfrak{Lgl}_n)$-module with the lowest weight vector $\xi^{\pm}$ given by

$$\xi^+ = e_1^{(1)} \otimes e_1^{(1)}, \qquad \xi^- = e_1^{(1)} \otimes e_2^{(1)} - q\, e_2^{(1)} \otimes e_1^{(1)}. \tag{4.2.9}$$

Denote

$$\mathscr{K}^{\theta}_{kj} := \big[K^{(1,1)}_{a\tilde{a}_i^0}(v, q^{2\theta} u_i^{(0)})\, K^{(1,1)}_{aa_i^0}(v, u_i^{(0)})\big]_{jk}, \qquad \mathscr{R}^{\theta}_{kj} := \big[R^{(1,1)}_{a\tilde{a}_i^0}(v, u_i^{(0)})\, R^{(1,1)}_{aa_i^0}(v, q^{-2\theta} u_i^{(0)})\big]_{jk},$$

where the matrix elements are taken with respect to the "a" space. Then $\mathscr{K}^{\theta}_{jk}\,\xi^{-\theta} = \mathscr{R}^{\theta}_{jk}\,\xi^{-\theta} = 0$ if $j > k$ and

$$\mathscr{K}^{-}_{jj}\,\xi^+ = \left(\delta_{j<n} + \delta_{jn}\,\frac{q^2 v - q^{-2} u_i^{(0)}}{v - u_i^{(0)}}\right)\xi^+, \qquad \mathscr{R}^{-}_{jj}\,\xi^+ = \left(\delta_{j1}\,\frac{q^{-2} v - q^{2} u_i^{(0)}}{v - u_i^{(0)}} + \delta_{j>1}\right)\xi^+,$$

$$\mathscr{K}^{+}_{jj}\,\xi^- = \left(\delta_{j<n-1} + \delta_{j\geq n-1}\,\frac{q v - q^{-1} u_i^{(0)}}{v - u_i^{(0)}}\right)\xi^-, \qquad \mathscr{R}^{+}_{jj}\,\xi^- = \left(\delta_{j\leq 2}\,\frac{q^{-1} v - q\, u_i^{(0)}}{v - u_i^{(0)}} + \delta_{j>2}\right)\xi^-.$$

For each $1 \leq j \leq m_0$ define vector $\xi_-^{(j)}$ recursively by $\xi_-^{(1)} := \xi^-$ and

$$\xi_-^{(j)} := e_1^{(1)} \otimes \xi_-^{(j-1)} \otimes e_2^{(1)} - q\, e_2^{(1)} \otimes \xi_-^{(j-1)} \otimes e_1^{(1)}.$$

We also set $\xi_+^{(j)} := \big(e_1^{(1)}\big)^{\otimes 2j}$. Then for each $1 \leq k \leq n-1$ we define a *level-k nested vacuum vector* by

$$\eta_{\pm}^{(k)} := \eta \otimes \xi_{\pm}^{(m_0)} \otimes \big(e_1^{(2)}\big)^{\otimes m_1} \otimes \cdots \otimes \big(e_1^{(k+1)}\big)^{\otimes m_k} \in (L^{(k)})^0, \tag{4.2.10}$$

where $\eta = \eta_{s_1} \otimes \cdots \otimes \eta_{s_\ell}$ is the lowest weight vector of $L^{(0)} = L$. We then denote the $(1,1)$-th matrix element of monodromy matrices (4.2.4), (4.2.6) by

$$\mathscr{a}^{(k)}(v; \boldsymbol{u}^{(0\ldots k-1)}) := \big[A_a^{(k)}(v; \boldsymbol{u}^{(0\ldots k-1)})\big]_{11}, \qquad \mathscr{d}^{(k)}(v; \boldsymbol{u}^{(0\ldots k-1)}) := \big[D_a^{(k)}(v; \boldsymbol{u}^{(0\ldots k-1)})\big]_{11}.$$

We will be interested in the action of these operators on $\eta_{\pm}^{(k)}$.

Lemma 4.2.3. *Vector $\eta_{-\theta}^{(k)}$ is lowest weight vector with respect to the action of the level-k monodromy matrix. The operators $\mathscr{a}^{(k)}(v; \boldsymbol{u}^{(0\ldots k-1)})$ and $\mathscr{d}^{(k)}(v; \boldsymbol{u}^{(0\ldots k-1)})$ act on $\eta_{-\theta}^{(k)}$ by multiplication* *with*

$$\text{for } k=1: \quad \prod_{i=1}^{\ell} \lambda_1^{(i)}(v)\,, \tag{4.2.11}$$

$$\text{for } 2\le k\le n-2: \quad \prod_{i=1}^{\ell} \lambda_k^{(i)}(v) \prod_{i=1}^{m_{k-1}} \frac{q^{-1}v - q\,u_i^{(k-1)}}{v-u_i^{(k-1)}}\,, \tag{4.2.12}$$

$$\text{for } k=n-1: \quad \prod_{i=1}^{\ell} \lambda_{n-1}^{(i)}(v) \prod_{i=1}^{m_{n-2}} \frac{q^{-1}v - q\,u_i^{(n-2)}}{v-u_i^{(n-2)}} \prod_{i=1}^{m_0} \frac{q^{\theta'}\,v - q^{-\theta'}\,u_i^{(0)}}{v-u_i^{(0)}}\,, \tag{4.2.13}$$

$$\text{for } k=n: \quad \prod_{i=1}^{\ell} \lambda_{n}^{(i)}(v) \prod_{i=1}^{m_{n-1}} \frac{q^{-1}v - q\,u_i^{(n-1)}}{v-u_i^{(n-1)}} \prod_{i=1}^{m_0} \frac{q^{2-\theta'}\,v - q^{\theta'-2}u_i^{(0)}}{v-u_i^{(0)}}\,, \tag{4.2.14}$$

and

$$\text{for } k=1: \quad \prod_{i=1}^{\ell} \lambda_{2n}^{(i)}(v)\,, \tag{4.2.15}$$

$$\text{for } 2\le k\le n-2: \quad \prod_{i=1}^{\ell} \lambda_{2n-k+1}^{(i)}(v) \prod_{i=1}^{m_{k-1}} \frac{q\,v - q^{2\kappa'-1}u_i^{(k-1)}}{v-q^{2\kappa'}u_i^{(k-1)}}\,, \tag{4.2.16}$$

$$\text{for } k=n-1: \quad \prod_{i=1}^{\ell} \lambda_{n+2}^{(i)}(v) \prod_{i=1}^{m_{n-2}} \frac{q\,v - q^{2\kappa'-1}u_i^{(n-2)}}{v-q^{2\kappa'}u_i^{(n-2)}} \prod_{i=1}^{m_0} \frac{q^{-\theta'}v - q^{\theta'}\,u_i^{(0)}}{v-u_i^{(0)}}\,, \tag{4.2.17}$$

$$\text{for } k=n: \quad \prod_{i=1}^{\ell} \lambda_{n+1}^{(i)}(v) \prod_{i=1}^{m_{n-1}} \frac{q\,v - q^{2\kappa'-1}u_i^{(n-1)}}{v-q^{2\kappa'}u_i^{(n-1)}} \prod_{i=1}^{m_0} \frac{q^{\theta'-2}\,v - q^{2-\theta'}\,u_i^{(0)}}{v-u_i^{(0)}}\,, \tag{4.2.18}$$

respectively. Here $\kappa' = \kappa - k + 1$ *and* $\theta' = \frac{1}{2}(1+\theta)$.

Proof. First, note that the level-k nested monodromy matrix may be written as

$$A_a^{(k)}(v;\boldsymbol{u}^{(0\ldots k-1)}) = \left(\prod_{i=m_0}^{1} K_{a\tilde a_i^0}^{(k,1)}(v, q^{2\theta}u_i^{(0)})\right) \\ \times \left(\prod_{i=1}^{m_0} K_{aa_i^0}^{(k,1)}(v,u_i^{(0)})\right) A_a^{(k)}(v) \left(\prod_{l=2}^{k}\prod_{i=1}^{m_{l-1}} R_{aa_i^{l-1}}^{(k,l)}(v,u_i^{(l-1)})\right).$$

In order to prove that $\eta_{-\theta}^{(k)}$ is a lowest weight vector, consider the action of the matrix elements $\left[A_a^{(k)}(v;\boldsymbol{u}^{(0\ldots k-1)})\right]_{ij}$ with $i\ge j$. It follows from (4.1.17–4.1.18) that, when acting on $\eta_+^{(k)}$, the R- and K-matrices become upper triangular matrices in the "a" auxiliary space. That is, for $i\ge j$,

$$\mathcal{E}_{j'i'}^{(l)}(u/v)\,e_1^{(l)} = \delta_{ij}\left(\delta_{i'1}\,\frac{q^{-1}v - q\,u}{v-u} + \delta_{i'>1}\right)e_1^{(l)},$$

$$(\mathcal{E}_{q^{-1},j'i'}^{(l)}(u/v))^{\bar\omega}\,e_1^{(l)} = \delta_{ij}\left(\delta_{i'\bar l}\,\frac{q\,v - q^{-1}u}{v-u} + \delta_{i'<\bar l}\right)e_1^{(l)},$$

where the primed notation is the same as for (4.1.18). Furthermore, as η is a lowest weight vector for L, the action of $A_a^{(k)}(v)$ is also upper triangular,

$$\left[A_a^{(k)}(v)\right]_{ij}\eta = \delta_{ij}\prod_{p=1}^{\ell}\lambda_k^{(p)}(v,c_p)\,\eta \qquad \text{for } i \geq j.$$

Therefore, taking a product of these matrices, the action of the level-k nested monodromy matrix will also be upper triangular, from which we conclude that $\eta_+^{(k)}$ is a lowest weight vector. The identities (4.2.11-4.2.14) may be found by

$$\begin{aligned}
\left[\prod_{i=1}^{m_{l-1}} R^{(k,l)}_{aa_i^{l-1}}(v,u_i^{(l-1)})\right]_{j1} (e_1^{(l)})^{\otimes m_{l-1}} &= \delta_{j1}\left(\delta_{kl}\prod_{i=1}^{m_{k-1}}\frac{q^{-1}v - q\,u_i^{(k-1)}}{v - u_i^{(k-1)}} + \delta_{k>l}\right)(e_1^{(l)})^{\otimes m_{l-1}},\\
\left[\prod_{i=m_0}^{1} K^{(k,1)}_{a\tilde{a}_i^0}(v,q^{2\theta}u_i^{(0)})\right]_{j1} (e_1^{(1)})^{\otimes m_0} &= \delta_{j1}\left(\delta_{kn}\prod_{i=1}^{m_0}\frac{q^{1-\theta}\,v - q^{\theta-1}u_i^{(0)}}{q^{-\theta}v - q^{\theta}u_i^{(0)}} + \delta_{k<n}\right)(e_1^{(1)})^{\otimes m_0},\\
\left[\prod_{i=1}^{m_0} K^{(k,1)}_{aa_i^0}(v,u_i^{(0)})\right]_{11} (e_1^{(1)})^{\otimes m_0} &= \left(\delta_{kn}\prod_{i=1}^{m_0}\frac{q\,v - q^{-1}u_i^{(0)}}{v - u_i^{(0)}} + \delta_{k<n}\right)(e_1^{(1)})^{\otimes m_0},
\end{aligned}$$

where the matrix elements are taken with respect to the the "a" space. Expressions (4.2.15-4.2.18) are obtained similarly. This concludes the proof in the symplectic case.

The orthogonal case follows by the same arguments and the fact that $\xi_-^{(m_0)}$ is a lowest weight vector with respect to the action of

$$\mathscr{K}^{[m_0]} := \left(\prod_{i=m_0}^{1} K^{(1,1)}_{a\tilde{a}_i^0}(v,q^2u_i^{(0)})\right)\left(\prod_{i=1}^{m_0} K^{(1,1)}_{aa_i^0}(v,u_i^{(0)})\right).$$

We will prove the latter by induction on m_0. The $m_0 = 1$ case has already been explained above. Let $s \geq 1$. We assume that $\xi_-^{(s)}$ is a lowest weight vector for $\mathscr{K}^{[s]}$ of weight $\lambda_i^{[s]}(v) = 1$ for $i < n-1$ and $\lambda_i^{[s]}(v) = \prod_{j=1}^{s}\frac{q\,v - q^{-1}u}{v-u}$ for $i = n-1, n$. We write the action of $\mathscr{K}^{[s+1]}$ on $\xi_-^{(s+1)}$ as

$$\begin{aligned}
\left[\mathscr{K}^{[s+1]}_{ij}\right]_{ij}\cdot\xi_-^{(s+1)} &= \sum_{b,c=1}^{n}(\mathcal{E}^{(1)}_{q^{-1},bi}(q^2u/v))^{\bar{\omega}}\left[\mathscr{K}^{[s]}\right]_{bc}(\mathcal{E}^{(1)}_{q^{-1},jc}(u/v))^{\bar{\omega}}\\
&\qquad\times\left(e_1^{(1)}\otimes\xi_-^{(s)}\otimes e_2^{(1)} - q e_2^{(1)}\otimes\xi_-^{(s)}\otimes e_1^{(1)}\right)\\
&= \sum_{\substack{b,c=1\\ i\leq b\leq c}}^{n}(\mathcal{E}^{(1)}_{q^{-1},bi}(q^2u/v))^{\bar{\omega}}\left[\mathscr{K}^{[s]}\right]_{bc}(\mathcal{E}^{(1)}_{q^{-1},jc}(u/v))^{\bar{\omega}}\cdot e_1^{(1)}\otimes\xi_-^{(s)}\otimes e_2^{(1)}\\
&\quad - q\sum_{\substack{b,c=1\\ b\leq c\leq j}}^{n}(\mathcal{E}^{(1)}_{q^{-1},bi}(q^2u/v))^{\bar{\omega}}\left[\mathscr{K}^{[s]}\right]_{bc}(\mathcal{E}^{(1)}_{q^{-1},jc}(u/v))^{\bar{\omega}}\cdot e_2^{(1)}\otimes\xi_-^{(s)}\otimes e_1^{(1)} \qquad (4.2.19)
\end{aligned}$$

since $\xi_-^{(s)}$ and $e_1^{(1)}$ are lowest weight vectors in their relevant representations. Observe that

$$
\begin{aligned}
(\mathcal{E}^{(1)}_{q^{-1},ji}(u/v))^{\bar\omega}\, e_1^{(1)} &= \delta_{ij}\left(\delta_{in}\frac{qv-q^{-1}u}{v-u}+\delta_{i<n}\right)e_1^{(1)} \\
&\qquad + \delta_{i<n}\delta_{jn}\frac{q^{i-j}(q-q^{-1})}{v/u-1}\, e_{n-i+1}^{(1)}
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathcal{E}^{(1)}_{q^{-1},ji}(u/v))^{\bar\omega}\, e_2^{(1)} &= \delta_{ij}\left(\delta_{i,n-1}\frac{qv-q^{-1}u}{v-u}+\delta_{i\neq n-1}\right)e_2^{(1)} - \delta_{in}\delta_{j,n-1}\frac{q^2-1}{u/v-1}\, e_1^{(1)} \\
&\qquad + \delta_{i<n-1}\delta_{j,n-1}\frac{q^{i-j}(q-q^{-1})}{v/u-1}\, e_{n-i+1}^{(1)}\,.
\end{aligned}
$$

Assume that $i>j$. It clear from above that $\mathscr{K}^{[s+1]}_{ij}\cdot\xi_-^{(s+1)}=0$ if $j<n-1$. Hence we only need to consider the case with $i=n$ and $j=n-1$. Then (4.2.19) becomes

$$
\begin{aligned}
\big[\mathscr{K}^{[s+1]}\big]_{n,n-1}\cdot\xi_-^{(s+1)} &= -\Bigg(\lambda_n^{[s]}(v)\,\frac{v-u}{q^{-1}v-qu}\cdot\frac{q^2-1}{u/v-1} \\
&\qquad - q\,\lambda_n^{[s]}(v)\,\frac{q^2-1}{q^2u/v-1}\cdot\frac{qv-q^{-1}u}{v-u}\Bigg)\, e_1^{(1)}\otimes\xi_-^{(s)}\otimes e_1^{(1)} = 0,
\end{aligned}
$$

as required. Next, assume that $i=j=n$. Then

$$
\begin{aligned}
\big[\mathscr{K}^{[s+1]}\big]_{nn}\cdot\xi_-^{(s+1)} &= \lambda_n^{[s]}(v)\left(\frac{v-u}{q^{-1}v-qu}\, e_1^{(1)}\otimes\xi_-^{(s)}\otimes e_2^{(1)} - q\,\frac{qv-q^{-1}u}{v-u}\, e_2^{(1)}\otimes\xi_-^{(s)}\otimes e_1^{(1)}\right) \\
&\qquad + q\,\lambda_{n-1}^{[s]}(v)\,\frac{q^2-1}{q^2u/v-1}\cdot\frac{q^{-1}(q-q^{-1})}{v/u-1}\, e_1^{(1)}\otimes\xi_-^{(s)}\otimes e_2^{(1)} \\
&= \lambda_n^{[s]}(v)\,\frac{qv-q^{-1}u}{v-u}\left(e_1^{(1)}\otimes\xi_-^{(s)}\otimes e_2^{(1)} - q\,e_2^{(1)}\otimes\xi_-^{(s)}\otimes e_1^{(1)}\right) \\
&= \lambda_n^{[s+1]}(v)\,\xi_-^{(s+1)}.
\end{aligned}
$$

In a similar way, for $i=j=n-1$, we find

$$
\begin{aligned}
\big[\mathscr{K}^{[s+1]}\big]_{n-1,n-1}\cdot\xi_-^{(s+1)} &= \lambda_{n-1}^{[s]}(v)\left(\frac{qv-q^{-1}u}{v-u}\, e_1^{(1)}\otimes\xi_-^{(s)}\otimes e_2^{(1)} - q^2\,\frac{v-u}{v-q^2u}\, e_2^{(1)}\otimes\xi_-^{(s)}\otimes e_1^{(1)}\right) \\
&\qquad - \lambda_n^{[s]}(v)\,\frac{q^{-1}(q-q^{-1})}{q^{-2}v/u-1}\cdot\frac{q^2-1}{u/v-1}\, e_2^{(1)}\otimes\xi_-^{(s)}\otimes e_1^{(1)} \\
&= \lambda_{n-1}^{[s]}(v)\,\frac{qv-q^{-1}u}{v-u}\left(e_1^{(1)}\otimes\xi_-^{(s)}\otimes e_2^{(1)} - q\,e_2^{(1)}\otimes\xi_-^{(s)}\otimes e_1^{(1)}\right) \\
&= \lambda_{n-1}^{[s+1]}(v)\,\xi_-^{(s+1)}.
\end{aligned}
$$

Lastly, when $i=j<n-1$ we obtain $\big[\mathscr{K}^{[s+1]}\big]_{ii}\cdot\xi_-^{(s+1)}=\xi_-^{(s+1)}$. Then, using the arguments similar to those in the symplectic case yields the wanted result. $\square$

4.2.2 Transfer matrices, Bethe vectors and Bethe equations

Recall the level-k monodromy matrices from Definition 4.2.1.

Definition 4.2.4. *We define level-*0 *transfer matrix by*

$$\tau(v) := \mathrm{tr}_a\, T_a(v) = \mathrm{tr}_a\, A_a^{(1)}(v) + \mathrm{tr}_a\, D_a^{(1)}(v). \tag{4.2.20}$$

For all $1 \le k \le n-1$ *we define level-*k *transfer matrices by*

$$\tau^{(k)}(v;\boldsymbol{u}^{(0\ldots k-1)}) := \mathrm{tr}_a\, A_a^{(k)}(v;\boldsymbol{u}^{(0\ldots k-1)})$$

and

$$\widetilde{\tau}^{(k)}(v;\boldsymbol{u}^{(0\ldots k-1)}) := \mathrm{tr}_a\, D_a^{(k)}(v;\boldsymbol{u}^{(0\ldots k-1)}).$$

Note that, as there is no equivalent of the symmetry relation that we used in previous chapters, it is necessary to keep track of both the A and D blocks of the monodromy matrix.

Next, for each level of nesting, $0 \le k \le n-1$, we introduce the creation operators for multiple excitations.

Definition 4.2.5. *We define level-*0 *creation operator by*

$$\mathscr{B}^{(0)}_{\boldsymbol{a}^{\tilde{0},0}}(\boldsymbol{u}^{(0)}) := \prod_{i=1}^{m_0} \beta_{\tilde{a}_i^0 a_i^0}(u_i^{(0)}) \tag{4.2.21}$$

where

$$\beta_{\tilde{a}_i^0 a_i^0}(u_i^{(0)}) := \sum_{j,k=1}^{n} q^{-j}\, t_{\bar{\jmath},n+k}(u_i^{(0)}) \otimes e_k^{(1)*} \otimes e_j^{(1)*} \in \mathrm{End}(L^{(0)}) \otimes V^{(1)*}_{\tilde{a}_i^0} \otimes V^{(1)*}_{a_i^0}. \tag{4.2.22}$$

For all $1 \le k \le n-1$ *we define level-*k *creation operators by*

$$\mathscr{B}^{(k)}_{\boldsymbol{a}^k}(\boldsymbol{u}^{(k)};\boldsymbol{u}^{(0\ldots k-1)}) := \prod_{i=1}^{m_k} B^{(k+1)}_{a_i^k}(u_i^{(k)};\boldsymbol{u}^{(0\ldots k-1)}) \tag{4.2.23}$$

where

$$B^{(k+1)}_{a_i^k}(u_i^{(k)};\boldsymbol{u}^{(0\ldots k-1)}) := \sum_{j=1}^{n-k} \big[A^{(k)}_{a_i^k}(u_i^{(k)};\boldsymbol{u}^{(0\ldots k-1)})\big]_{1,1+j} \otimes e_j^{(k+1)*} \in \mathrm{End}(L^{(k)}) \otimes V^{(k+1)*}_{a_i^k}. \tag{4.2.24}$$

Recall the notion of nested vacuum vector $\eta_\pm^{(k)}$, viz. (4.2.10). The level-$(n-1)$ nested vacuum vector is our reference state for constructing the (off-shell) Bethe vectors.

Definition 4.2.6. *For all $1 \leq k \leq n-1$ we define level-k Bethe vectors by*

$$\Phi_\theta^{(k)}(\boldsymbol{u}^{(k\dots n-1)};\boldsymbol{u}^{(0\dots k-1)}) := \prod_{i=k}^{n-1} \mathscr{B}^{(i)}_{\boldsymbol{a}^i}(\boldsymbol{u}^{(i)};\boldsymbol{u}^{(0\dots i-1)}) \cdot \eta_{-\theta}^{(n-1)}. \tag{4.2.25}$$

The level-0 Bethe vector is defined by

$$\Phi_\theta^{(0)}(\boldsymbol{u}^{(0\dots n-1)}) := \mathscr{B}^{(0)}_{\boldsymbol{a}^{\tilde{0},0}}(\boldsymbol{u}^{(0)})\, \Phi_\theta^{(1)}(\boldsymbol{u}^{(1\dots n-1)};\boldsymbol{u}^{(0)}). \tag{4.2.26}$$

Note that vector $\Phi_\theta^{(k)}(\boldsymbol{u}^{(k\dots n-1)};\boldsymbol{u}^{(0\dots k-1)})$ is an element of the level-k quantum space $L^{(k)}$ and has $\boldsymbol{u}^{(0\dots k-1)}$ and $\boldsymbol{c}$ as its free parameters.

For $0 \leq k \leq n-1$ set $\mathfrak{S}_{\boldsymbol{m}_{k\dots n-1}} := \mathfrak{S}_{m_k} \times \cdots \times \mathfrak{S}_{m_{n-1}}$. For any $\sigma^{(l)} \in \mathfrak{S}_{m_l}$ with $k \leq l \leq n-1$ define an action of $\mathfrak{S}_{\boldsymbol{m}_{k\dots n-1}}$ on $\Phi_\theta^{(k)}(\boldsymbol{u}^{(k\dots n-1)};\boldsymbol{u}^{(0\dots k-1)})$ by

$$\sigma^{(l)} : \boldsymbol{u}^{(k\dots n-1)} \mapsto \boldsymbol{u}^{(k\dots n-1)}_{\sigma^{(l)}} := (\boldsymbol{u}^{(k)},\dots,\boldsymbol{u}^{(l)}_{\sigma^{(l)}},\dots,\boldsymbol{u}^{(n-1)}\}, \quad \boldsymbol{u}^{(l)}_{\sigma^{(l)}} := (u^{(l)}_{\sigma^{(l)}(1)},\dots,u^{(l)}_{\sigma^{(l)}(m_l)}).$$

For further convenience we set $\sigma_j^{(l)} \in \mathfrak{S}_{m_l}$ to be the j-cycle such that

$$\boldsymbol{u}^{(l)}_{\sigma_j^{(l)}} = (u_j^{(l)}, u_{j+1}^{(l)},\dots,u_{m_l}^{(l)},u_1^{(l)},\dots,u_{j-1}^{(l)}).$$

We will also make use of the notation

$$\boldsymbol{u}^{(l)}_{\sigma_j^{(l)},u_j^{(l)}\to v} := (v, u_{j+1}^{(l)},\dots,u_{m_l}^{(l)},u_1^{(l)},\dots,u_{j-1}^{(l)}).$$

Lemma 4.2.7. *Bethe vector $\Phi_\theta^{(k)}(\boldsymbol{u}^{(k\dots n-1)};\boldsymbol{u}^{(0\dots k-1)})$ is invariant under the action of $\mathfrak{S}_{\boldsymbol{m}_{k\dots n-1}}$.*

Proof. This follows using standard arguments, the fact that $\check{R}$-matrices act on $\eta_{-\theta}^{(n-1)}$ by 1, and relations

$$\beta_{\tilde{a}_i^0 a_i^0}(u_i^{(0)})\,\beta_{\tilde{a}_{i+1}^0 a_{i+1}^0}(u_{i+1}^{(0)}) = \beta_{\tilde{a}_i^0 a_i^0}(u_{i+1}^{(0)})\,\beta_{\tilde{a}_{i+1}^0 a_{i+1}^0}(u_i^{(0)})\,(\check{R}^{(1,1)}_{a_i^0 a_{i+1}^0}(u_i^{(0)},u_{i+1}^{(0)}))^{-1}\check{R}^{(1,1)}_{\tilde{a}_i^0\tilde{a}_{i+1}^0}(u_i^{(0)},u_{i+1}^{(0)})$$

and

$$\begin{aligned} &B^{(k+1)}_{a_i^k}(u_i^{(k)};\boldsymbol{u}^{(0\dots k-1)})B^{(k+1)}_{a_{i+1}^k}(u_{i+1}^{(k)};\boldsymbol{u}^{(0\dots k-1)}) \\ &\quad\equiv B^{(k+1)}_{a_i^k}(u_{i+1}^{(k)};\boldsymbol{u}^{(0\dots k-1)})B^{(k+1)}_{a_{i+1}^k}(u_i^{(k)};\boldsymbol{u}^{(0\dots k-1)})\check{R}^{(k+1,k+1)}_{a_i^k a_{i+1}^k}(u_i^{(k)},u_{i+1}^{(k)}) \end{aligned}$$

which follow from (4.1.29) and (4.1.33); here $\equiv$ denotes equality of operators in the space $L^{(k)}$. □

The following result is the solution of the $U_q(\mathfrak{gl}_n)$ subsystem. The method is well-known—see e.g. [BR08]—and follows along the same line as the rational case, which was given in Section 2.1, and so we omit the details here.

Theorem 4.2.8. *Bethe vector $\Phi^{(1)}_\theta(\boldsymbol{u}^{(1\dots n-1)};\boldsymbol{u}^{(0)})$ is an eigenvector of $\tau^{(1)}(v;\boldsymbol{u}^{(0)})$ with the eigenvalue*

$$
\begin{aligned}
\Lambda^{(1)}(v;\boldsymbol{u}^{(1\dots n-1)};\boldsymbol{u}^{(0)}) := &\prod_{i=1}^{m_1}\frac{q\,v-q^{-1}u_i^{(1)}}{v-u_i^{(1)}}\prod_{i=1}^{\ell}\lambda_1^{(i)}(v)\\
&+\sum_{k=2}^{n-2}\prod_{i=1}^{m_{k-1}}\frac{q^{-1}v-q\,u_i^{(k-1)}}{v-u_i^{(k-1)}}\prod_{i=1}^{m_k}\frac{q\,v-q^{-1}u_i^{(k)}}{v-u_i^{(k)}}\prod_{i=1}^{\ell}\lambda_k^{(i)}(v)\\
&+\prod_{i=1}^{m_{n-2}}\frac{q^{-1}v-q\,u_i^{(n-2)}}{v-u_i^{(n-2)}}\prod_{i=1}^{m_{n-1}}\frac{q\,v-q^{-1}\,u_i^{(n-1)}}{v-u_i^{(n-1)}}\prod_{i=1}^{m_0}\frac{q^{\theta'}\,v-q^{-\theta'}\,u_i^{(0)}}{v-u_i^{(0)}}\prod_{i=1}^{\ell}\lambda_{n-1}^{(i)}(v)\\
&+\prod_{i=1}^{m_0}\frac{q^{2-\theta'}v-q^{-2+\theta'}u_i^{(0)}}{v-u_i^{(0)}}\prod_{i=1}^{m_{n-1}}\frac{q^{-1}v-q\,u_i^{(n-1)}}{v-u_i^{(n-1)}}\prod_{i=1}^{\ell}\lambda_n^{(i)}(v)
\end{aligned}
\tag{4.2.27}
$$

and an eigenvector of $\widetilde{\tau}^{(1)}(v;\boldsymbol{u}^{(0)})$ with the eigenvalue

$$
\begin{aligned}
\widetilde{\Lambda}^{(1)}(v;\boldsymbol{u}^{(1\dots n-1)};\boldsymbol{u}^{(0)}) := &\prod_{i=1}^{m_1}\frac{v-q^{2\kappa}u_i^{(1)}}{q\,v-q^{2\kappa-1}u_i^{(1)}}\prod_{i=1}^{\ell}\lambda_{2n}^{(i)}(v)\\
&+\sum_{k=2}^{n-2}\prod_{i=1}^{m_{k-1}}\frac{q\,v-q^{2\kappa-2k+1}u_i^{(k-1)}}{v-q^{2\kappa-2k+2}u_i^{(k-1)}}\prod_{i=1}^{m_k}\frac{v-q^{2\kappa-2k+2}u_i^{(k)}}{q\,v-q^{2\kappa-2k+1}u_i^{(k)}}\prod_{i=1}^{\ell}\lambda_{2n-k+1}^{(i)}(v)\\
&+\prod_{i=1}^{m_{n-2}}\frac{q\,v-q^{3-2\theta}u_i^{(n-2)}}{v-q^{4-2\theta}u_i^{(n-2)}}\prod_{i=1}^{m_{n-1}}\frac{v-q^{4-2\theta}\,u_i^{(n-1)}}{q\,v-q^{3-2\theta}u_i^{(n-1)}}\prod_{i=1}^{m_0}\frac{q^{-\theta'}\,v-q^{\theta'}\,u_i^{(0)}}{v-u_i^{(0)}}\prod_{i=1}^{\ell}\lambda_{n+2}^{(i)}(v)\\
&+\prod_{i=1}^{m_{n-1}}\frac{q\,v-q^{1-2\theta}u_i^{(n-1)}}{v-q^{2-2\theta}u_i^{(n-1)}}\prod_{i=1}^{m_0}\frac{q^{-2+\theta'}\,v-q^{2-\theta'}\,u_i^{(0)}}{v-u_i^{(0)}}\prod_{i=1}^{\ell}\lambda_{n+1}^{(i)}(v)
\end{aligned}
\tag{4.2.28}
$$

provided

$$
\operatorname*{Res}_{v\to u_j^{(k)}}\Lambda^{(1)}(v;\boldsymbol{u}^{(1\dots n-1)};\boldsymbol{u}^{(0)})=0\quad\text{for } 1\le k\le n-1,\ 1\le j\le m_k. \tag{4.2.29}
$$

Proof. Step 1. $\Phi^{(1)}_\theta(\boldsymbol{u}^{(1\dots n-1)};\boldsymbol{u}^{(0)})$ is an eigenvector of $\tau^{(1)}(v,\boldsymbol{u}^{(0)})$. We rewrite the exchange relations (4.1.31) and (4.1.32) as

$$
\begin{aligned}
\mathcal{a}^{(k)}(v)\,B^{(k+1)}_{a_i^k}(u_i^{(k)}) = &\frac{q\,v-q^{-1}u_i^{(k)}}{v-u_i^{(k)}}B^{(k+1)}_{a_i^k}(u_i^{(k)})\,\mathcal{a}^{(k)}(v)\\
&-\frac{v/u_i^{(k)}}{v-u_i^{(k)}}\operatorname*{Res}_{w\to u_i^{(k)}}\frac{q\,w-q^{-1}u_i^{(k)}}{w-u_i^{(k)}}B^{(k+1)}_{a_i^k}(v)\,\mathcal{a}^{(k)}(w),\\
A^{(k+1)}_a(v)\,B^{(k+1)}_{a_i^k}(u_i^{(k)}) = &B^{(k+1)}_{a_i^k}(u_i^{(k)})\,A^{(k+1)}_a(v)\,R^{(k+1)}_{aa_i^k}(v,u_i^{(k)})\\
&-\frac{v/u_i^{(k)}}{v-u_i^{(k)}}\operatorname*{Res}_{w\to u_i^{(k)}}B^{(k+1)}_{a_i^k}(v)A^{(k+1)}_a(w)\,R^{(k+1,k+1)}_{aa_i^k}(w,u_i^{(k)}).
\end{aligned}
$$

Then, using the usual symmetry arguments for the Bethe vector, viz. Lemma 4.2.7, we obtain

$$\begin{aligned}
&\tau^{(1)}(v;\boldsymbol{u}^{(0)})\,\Phi_\theta^{(1)}(\boldsymbol{u}^{(1\dots n-1)};\boldsymbol{u}^{(0)})\\
&\quad=\mathscr{B}^{(1)}_{\boldsymbol{a}^1}(\boldsymbol{u}^{(1)};\boldsymbol{u}^{(0)})\Bigg(\prod_{i=1}^{m_1}\frac{qv-q^{-1}u_i^{(1)}}{v-u_i^{(1)}}\,a^{(1)}(v;\boldsymbol{u}^{(0)})+\tau^{(2)}(v;\boldsymbol{u}^{(0,1)})\Bigg)\Phi_\theta^{(2)}(\boldsymbol{u}^{(2\dots n-1)};\boldsymbol{u}^{(0,1)})\\
&\qquad-\sum_{j=1}^{m_1}\frac{v/u_j^{(1)}}{v-u_j^{(1)}}\mathscr{B}^{(1)}_{\boldsymbol{a}^1}(\boldsymbol{u}^{(1)}_{\sigma_j^{(1)},u_j^{(1)}\to v};\boldsymbol{u}^{(0)})\\
&\qquad\times\operatorname*{Res}_{w\to u_j^{(1)}}\Bigg(\prod_{i=1}^{m_1}\frac{qw-q^{-1}u_i^{(1)}}{w-u_i^{(1)}}\,a^{(1)}(w;\boldsymbol{u}^{(0)})+\tau^{(2)}(w;\boldsymbol{u}^{(0,1)}_{\sigma_j^{(1)},u_j^{(1)}\to v})\Bigg)\Phi_\theta^{(2)}(\boldsymbol{u}^{(2\dots n-1)};\boldsymbol{u}^{(0,1)}_{\sigma_j^{(1)}}).
\end{aligned}$$

Proceeding in the same way and using Lemma 4.2.3 we find

$$\tau^{(1)}(v;\boldsymbol{u}^{(0)})\,\Phi_\theta^{(1)}(\boldsymbol{u}^{(1\dots n-1)};\boldsymbol{u}^{(0)})=\Lambda^{(1)}(v;\boldsymbol{u}^{(1\dots n-1)};\boldsymbol{u}^{(0)})\,\Phi_\theta^{(1)}(\boldsymbol{u}^{(1\dots n-1)};\boldsymbol{u}^{(0)})$$

provided (4.2.29) holds, as required.
Step 2. $\Phi_\theta^{(1)}(\boldsymbol{u}^{(1\dots n-1)};\boldsymbol{u}^{(0)})$ *is an eigenvector of* $\widetilde{\tau}^{(1)}(v,\boldsymbol{u}^{(0)})$. It follows from Lemma 4.2.2 that transfer matrices $\tau^{(1)}(v;\boldsymbol{u}^{(0)})$ and $\widetilde{\tau}^{(1)}(v;\boldsymbol{u}^{(0)})$ form a family of commutative operators in the space $L^{(1)}$. They can thus be diagonalized simultaneously. Assuming (4.2.29) holds, it is sufficient to focus on the wanted terms in the exchange relations. In particular, it follows from (4.2.8) that

$$d^{(k)}(v)\,B^{(k+1)}_{a_i^k}(u_i^{(k)})=\frac{v-q^{2\kappa-2k+2}u_i^{(k)}}{qv-q^{2\kappa-2+1}u_i^{(k)}}\,B^{(k+1)}_{a_i^k}(u_i^{(k)})\,d^{(k)}(v)+UWT,$$

$$D^{(k+1)}_a(v)\,B^{(k+1)}_{a_i^k}(u_i^{(k)})=B^{(k+1)}_{a_i^k}(u_i^{(k)})\,R^{(k+1)}_{q^{-1},aa_i^k}(v,q^{2\kappa-2k+2}u_i^{(k)})\,D^{(k+1)}_a(v)+UWT,$$

where UWT denotes the unwanted terms. The eigenvalue (4.2.28) now follows by Lemma 4.2.3 and the standard arguments. □

Having solved the nested system, we now give the solution to the full system below.

Theorem 4.2.9. *Bethe vector* $\Phi_\theta^{(0)}(\boldsymbol{u}^{(0\dots n-1)})$ *is an eigenvector of* $\tau(v)$ *with eigenvalue*

$$\Lambda(v;\boldsymbol{u}^{(0\dots n-1)}):=\Lambda^{(1)}(v;\boldsymbol{u}^{(1\dots n-1)};\boldsymbol{u}^{(0)})+\widetilde{\Lambda}^{(1)}(v;\boldsymbol{u}^{(1\dots n-1)};\boldsymbol{u}^{(0)})\tag{4.2.30}$$

provided

$$\operatorname*{Res}_{v\to u_j^{(0)}}\Lambda(v;\boldsymbol{u}^{(0\dots n-1)})=0\quad\textit{for }\ 1\le j\le m_0,\quad\textit{and}\tag{4.2.31}$$

$$\operatorname*{Res}_{v\to u_j^{(k)}}\Lambda^{(1)}\big(v;\boldsymbol{u}^{(1\dots n-1)};\boldsymbol{u}^{(0)}\big)=0\quad\textit{for }\ 1\le j\le m_k,\ 1\le k\le n-1.\tag{4.2.32}$$

Proof. Using (4.1.11) and (4.1.14)–(4.1.16), we deduce $R^{(1,1)}_{21}(u,v) = R^{(1,1)}_{q^{-1},12}(v,u)$ and

$$(K^{(1,1)}_{12}(u,q^{2\kappa}v))^{-1} = K^{(1,1)}_{21}(v,q^{2\theta}u) = K^{(1,1)}_{q^{-1},12}(u,q^{-2\theta}v).$$

These relations allow us to rewrite (4.1.24) and (4.1.25) as

$$\begin{aligned}
A^{\pm}_1(v)B^{\pm}_2(u) &= R^{(1,1)}_{q^{-1},12}(v,u)B^{\pm}_2(u)A^{\pm}_1(v)K^{(1,1)}_{12}(v,q^{2\theta}u) \\
&\quad - \frac{1}{v-u}\operatorname*{Res}_{w\to u} R^{(1,1)}_{q^{-1},12}(w,u)B^{\pm}_1(v)A^{\pm}_2(u)K^{(1,1)}_{12}(w,q^{2\theta}u), \\
D^{\pm}_1(v)B^{\pm}_2(u) &= K^{(1,1)}_{q^{-1},12}(v,q^{-2\theta}u)B^{\pm}_2(u)D^{\pm}_1(v)R^{(1,1)}_{12}(v,u) \\
&\quad - \frac{1}{v-u}\operatorname*{Res}_{w\to u} K^{(1,1)}_{q^{-1},12}(w,q^{-2\theta}u)B^{\pm}_1(v)D^{\pm}_2(u)R^{(1,1)}_{12}(w,u).
\end{aligned}$$

Then, using (4.1.14), (4.2.22) and replacing the A and D operators with their images in $\mathrm{End}(L^{(0)})$, we obtain

$$\begin{aligned}
A^{(1)}_a(v)\,\beta_{\tilde a^0_i a^0_i}(u^{(0)}_i) &= \beta_{\tilde a^0_i a^0_i}(u^{(0)}_i)\,K^{(1,1)}_{aa^0_i}(v,u^{(0)}_i)A^{(1)}_a(v)K^{(1,1)}_{a\tilde a^0_i}(v,q^{2\theta}u^{(0)}_i) \\
&\quad - \frac{1}{v-u^{(0)}_i}\,\beta_{\tilde a^0_i a^0_i}(v)\operatorname*{Res}_{w\to u^{(0)}_i} K^{(1,1)}_{aa^0_i}(w,u^{(0)}_i)A^{(1)}_a(w)K^{(1,1)}_{a\tilde a^0_i}(w,q^{2\theta}u^{(0)}_i), \\
D^{(1)}_a(v)\,\beta_{\tilde a^0_i a^0_i}(u^{(0)}_i) &= \beta_{\tilde a^0_i a^0_i}(u^{(0)}_i)\,R^{(1,1)}_{aa^0_i}(v,q^{-2\theta}u^{(0)}_i)D^{(1)}_a(v)R^{(1,1)}_{a\tilde a^0_i}(v,u^{(0)}_i) \\
&\quad - \frac{1}{v-u^{(0)}_i}\,\beta_{\tilde a^0_i a^0_i}(v)\operatorname*{Res}_{w\to u^{(0)}_i} R^{(1,1)}_{aa^0_i}(w,q^{-2\theta}u^{(0)}_i)D^{(1)}_a(v)R^{(1,1)}_{a\tilde a^0_i}(w,u^{(0)}_i).
\end{aligned}$$

The relations above together with Lemma 4.2.7 and the standard symmetry arguments imply that

$$\begin{aligned}
&\tau(v)\,\Phi^{(0)}_\theta(\boldsymbol{u}^{(0\ldots n-1)}) \\
&\quad = \mathscr{B}^{(0)}_{\boldsymbol{a}^{\tilde 0,0}}(\boldsymbol{u}^{(0)})\Big(\tau^{(1)}(v;\boldsymbol{u}^{(0)}) + \widetilde\tau^{(1)}(v;\boldsymbol{u}^{(0)})\Big)\,\Phi^{(1)}_\theta(\boldsymbol{u}^{(1\ldots n-1)};\boldsymbol{u}^{(0)}) \\
&\qquad - \sum_{j=1}^{m_0} \frac{1}{v-u^{(0)}_j}\,\mathscr{B}^{(0)}_{\boldsymbol{a}^{\tilde 0,0}}(\boldsymbol{u}^{(0)}_{\sigma^{(0)}_j,u^{(0)}_j\to v}) \\
&\qquad\quad \times \operatorname*{Res}_{w\to u^{(0)}_j}\Big(\tau^{(1)}(w;\boldsymbol{u}^{(0)}_{\sigma^{(0)}_j}) + \widetilde\tau^{(1)}(w;\boldsymbol{u}^{(0)}_{\sigma^{(0)}_j})\Big)\,\Phi^{(1)}_\theta(\boldsymbol{u}^{(1\ldots n-1)};\boldsymbol{u}^{(0)}_{\sigma^{(0)}_j}).
\end{aligned}$$

Theorem (4.2.8) allows us to replace $\tau^{(1)}(w;\boldsymbol{u}^{(0)}_{\sigma^{(0)}_j})$ and $\widetilde\tau^{(1)}(w;\boldsymbol{u}^{(0)}_{\sigma^{(0)}_j})$ with their eigenvalues, provided

(4.2.32) holds, giving

$$\begin{aligned}
&\tau(v)\,\Phi^{(0)}_{\theta}(\boldsymbol{u}^{(0\ldots n-1)})\\
&\quad=\mathscr{B}^{(0)}_{\boldsymbol{a}^{\tilde{0},0}}(\boldsymbol{u}^{(0)})\,\Lambda(v;\boldsymbol{u}^{(0\ldots n-1)})\,\Phi^{(1)}_{\theta}(\boldsymbol{u}^{(1\ldots n-1)};\boldsymbol{u}^{(0)})\\
&\qquad-\sum_{j=1}^{m_0}\frac{1}{v-u^{(0)}_j}\mathscr{B}^{(0)}_{\boldsymbol{a}^{\tilde{0},0}}(\boldsymbol{u}^{(0)}_{\sigma^{(0)}_j,u^{(0)}_j\to v})\operatorname*{Res}_{w\to u^{(0)}_j}\Lambda(w;\boldsymbol{u}^{(0)}_{\sigma^{(0)}_j},\boldsymbol{u}^{(1\ldots n-1)})\,\Phi^{(1)}_{\theta}(\boldsymbol{u}^{(1\ldots n-1)};\boldsymbol{u}^{(0)}_{\sigma^{(0)}_j}).
\end{aligned}$$

Noting that, from its definition and the exact forms of the $\Lambda^{(1)}$ and $\widetilde{\Lambda}^{(1)}$ from Theorem 4.2.8, we have $\Lambda(w;\boldsymbol{u}^{(0)}_{\sigma^{(0)}_j},\boldsymbol{u}^{(1\ldots n-1)})=\Lambda(w;\boldsymbol{u}^{(0\ldots n-1)})$. Therefore, (4.2.31) implies that each term in the sum on the second line individually vanishes, and we are left with the desired result. □

Remark 4.2.10 (i). The equations (4.2.32) are Bethe equations for a $U_q(\mathfrak{gl}_n)$-symmetric spin chain, with an additional factor when $j=n-1$ due to level-0 excitations. For convenience, set $u^{(n)}_j:=u^{(0)}_j$ and $m_n:=m_0$. Then the explicit form of the equations (4.2.32) and (4.2.31) in the symplectic case is

$$\prod_{i=1}^{\ell}\frac{\lambda^{(i)}_1(u^{(1)}_j)}{\lambda^{(i)}_2(u^{(1)}_j)}=\prod_{\substack{i=1\\i\neq j}}^{m_1}\frac{q^{-1}u^{(1)}_j-q\,u^{(1)}_i}{q\,u^{(1)}_j-q^{-1}u^{(1)}_i}\prod_{i=1}^{m_2}\frac{q\,u^{(1)}_j-q^{-1}u^{(2)}_i}{u^{(1)}_j-u^{(2)}_i},\tag{4.2.33}$$

$$\prod_{i=1}^{\ell}\frac{\lambda^{(i)}_k(u^{(k)}_j)}{\lambda^{(i)}_{k+1}(u^{(k)}_j)}=\prod_{i=1}^{m_{k-1}}\frac{u^{(k)}_j-u^{(k-1)}_i}{q^{-1}u^{(k)}_j-q\,u^{(k-1)}_i}\prod_{\substack{i=1\\i\neq j}}^{m_k}\frac{q^{-1}u^{(k)}_j-q\,u^{(k)}_i}{q\,u^{(k)}_j-q^{-1}u^{(k)}_i}\prod_{i=1}^{m_{k+1}}\frac{q\,u^{(k)}_j-q^{-1}u^{(k+1)}_i}{u^{(k)}_j-u^{(k+1)}_i},\tag{4.2.34}$$

$$\prod_{i=1}^{\ell}\frac{\lambda^{(i)}_{n-1}(u^{(n-1)}_j)}{\lambda^{(i)}_n(u^{(n-1)}_j)}=\prod_{i=1}^{m_{n-2}}\frac{u^{(n-1)}_j-u^{(n-2)}_i}{q^{-1}u^{(n-1)}_j-q\,u^{(n-2)}_i}\prod_{\substack{i=1\\i\neq j}}^{m_{n-1}}\frac{q^{-1}u^{(n-1)}_j-q\,u^{(n-1)}_i}{q\,u^{(n-1)}_j-q^{-1}u^{(n-1)}_i}\prod_{i=1}^{m_n}\frac{q^2\,u^{(n-1)}_j-q^{-2}u^{(n)}_i}{u^{(n-1)}_j-u^{(n)}_i},\tag{4.2.35}$$

$$\prod_{i=1}^{\ell}\frac{\lambda^{(i)}_n(u^{(n)}_j)}{\lambda^{(i)}_{n+1}(u^{(n)}_j)}=\prod_{i=1}^{m_{n-1}}\frac{u^{(n)}_j-u^{(n-1)}_i}{q^{-2}u^{(n)}_j-q^2u^{(n-1)}_i}\prod_{\substack{i=1\\i\neq j}}^{m_n}\frac{q^{-2}\,u^{(n)}_j-q^2u^{(n)}_i}{q^2\,u^{(n)}_j-q^{-2}u^{(n)}_i}\tag{4.2.36}$$

for $2\le k\le n-2$ and all allowed j, and weights given by (4.1.36).

(ii). In the orthogonal case, the Bethe equations for $k=1,\ldots,n-3$ are identical to the symplectic case. For $k=n-2$ and $n-1$, the equations are replaced by, respectively,

$$\begin{aligned}\prod_{i=1}^{\ell}\frac{\lambda^{(i)}_{n-2}(u^{(n-2)}_j)}{\lambda^{(i)}_{n-1}(u^{(n-2)}_j)}&=\prod_{i=1}^{m_{n-3}}\frac{u^{(n-2)}_j-u^{(n-3)}_i}{q^{-1}u^{(n-2)}_j-q\,u^{(n-3)}_i}\prod_{\substack{i=1\\i\neq j}}^{m_{n-2}}\frac{q^{-1}u^{(n-2)}_j-q\,u^{(n-2)}_i}{q\,u^{(n-2)}_j-q^{-1}u^{(n-2)}_i}\\
&\quad\times\prod_{i=1}^{m_{n-1}}\frac{q\,u^{(n-2)}_j-q^{-1}u^{(n-1)}_i}{u^{(n-2)}_j-u^{(n-1)}_i}\prod_{i=1}^{m_n}\frac{q\,u^{(n-2)}_j-q^{-1}u^{(n)}_i}{u^{(n-2)}_j-u^{(n)}_i},\end{aligned}\tag{4.2.37}$$

and

$$\prod_{i=1}^{\ell} \frac{\lambda_{n-1}^{(i)}(u_j^{(n-1)})}{\lambda_n^{(i)}(u_j^{(n-1)})} = \prod_{i=1}^{m_{n-2}} \frac{u_j^{(n-1)} - u_i^{(n-2)}}{q^{-1}u_j^{(n-1)} - q u_i^{(n-2)}} \prod_{\substack{i=1\\ i\neq j}}^{m_{n-1}} \frac{q^{-1}u_j^{(n-1)} - q u_i^{(n-1)}}{q u_j^{(n-1)} - q^{-1}u_i^{(n-1)}}, \tag{4.2.38}$$

for all allowed j. For the level-0 Bethe equations, however, the eigenvalue contains four poles at each Bethe root, rather than two. Through use of the identity (see Remark 6.6 in [GRW20]),

$$\frac{\lambda_{n+2}(u)}{\lambda_n(u)} = \frac{\lambda_{n+1}(u)}{\lambda_{n-1}(u)},$$

the resulting expression may be factorised to give the following Bethe equations, for $1 \leq j \leq m_n$,

$$\begin{aligned}&\left(\prod_{i=1}^{m_{n-2}} \frac{q^{-1}u_j^{(n)} - q u_i^{(n-2)}}{u_j^{(n)} - u_i^{(n-2)}} \prod_{i=1}^{m_{n-1}} \frac{q u_j^{(n)} - q^{-1}u_i^{(n-1)}}{q^{-1}u_j^{(n)} - q u_i^{(n-1)}} + \prod_{i=1}^{\ell} \frac{\lambda_n^{(i)}(u_j^{(n)})}{\lambda_{n-1}^{(i)}(u_j^{(n)})}\right)\\ &\quad\times\left(\prod_{\substack{i=1\\ i\neq j}}^{m_n} \frac{q u_j^{(n)} - q^{-1}u_i^{(n)}}{q^{-1}u_j^{(n)} - q u_i^{(n)}} \prod_{i=1}^{m_{n-2}} \frac{q^{-1}u_j^{(n)} - q u_i^{(n-2)}}{u_j^{(n)} - u_i^{(n-2)}} - \prod_{i=1}^{\ell} \frac{\lambda_{n+1}^{(i)}(u_j^{(n)})}{\lambda_{n-1}^{(i)}(u_j^{(n)})}\right) = 0.\end{aligned} \tag{4.2.39}$$

Observe that setting the first factor to zero is exactly equivalent to (4.2.38), the level-$(n-1)$ set of equations, noting that the sign discrepancy is due to the product in (4.2.38) excluding the $i = j$ index. This factorisation is due to an automorphism of the Dynkin diagram of type D_n, which exchanges the two branching nodes. This symmetry of Dynkin diagram is broken by our nesting procedure, and so we obtained only a single set of Bethe equations for the level-$(n-1)$ Bethe roots. Extending this to the level-0, we set the right-hand factor above to zero to give the level-0 Bethe equations

$$\prod_{i=1}^{\ell} \frac{\lambda_{n-1}^{(i)}(u_j^{(n)}, c_i)}{\lambda_{n+1}^{(i)}(u_j^{(n)}, c_i)} = \prod_{i=1}^{m_{n-2}} \frac{u_j^{(n)} - u_i^{(n-2)}}{q^{-1}u_j^{(n)} - q u_i^{(n-2)}} \prod_{\substack{i=1\\ i\neq j}}^{m_n} \frac{q^{-1}u_j^{(n)} - q u_i^{(n)}}{q u_j^{(n)} - q^{-1}u_i^{(n)}} \tag{4.2.40}$$

for all allowed j.

(iii). Rather than taking (4.2.31) and (4.2.32) separately, we could instead attempt to recover the Bethe equations from the condition $\operatorname*{Res}_{v\to u_j^{(k)}} \Lambda(v; \boldsymbol{u}^{(0\ldots n-1)}) = 0$ for all Bethe roots $u_j^{(k)}$, $0 \leq k \leq n-1$. As one might expect, this turns out to be directly equivalent to (4.2.32) for $1 \leq k \leq n-2$, however, in the $k = n-1$ case we obtain a factorisation identical to (4.2.39),

$$\begin{aligned}&\left(\prod_{i=1}^{m_{n-2}} \frac{q^{-1}u_j^{(n-1)} - q u_i^{(n-2)}}{u_j^{(n-1)} - u_i^{(n-2)}} \prod_{\substack{i=1\\ i\neq j}}^{m_{n-1}} \frac{q u_j^{(n-1)} - q^{-1}u_i^{(n-1)}}{q^{-1}u_j^{(n-1)} - q u_i^{(n-1)}} - \prod_{i=1}^{\ell} \frac{\lambda_n^{(i)}(u_j^{(n-1)})}{\lambda_{n-1}^{(i)}(u_j^{(n-1)})}\right)\\ &\quad\times\left(\prod_{i=1}^{m_n} \frac{q u_j^{(n-1)} - q^{-1}u_i^{(n)}}{q^{-1}u_j^{(n-1)} - q u_i^{(n)}} \prod_{i=1}^{m_{n-2}} \frac{q^{-1}u_j^{(n-1)} - q u_i^{(n-2)}}{u_j^{(n-1)} - u_i^{(n-2)}} + \prod_{i=1}^{\ell} \frac{\lambda_{n+1}^{(i)}(u_j^{(n-1)})}{\lambda_{n-1}^{(i)}(u_j^{(n-1)})}\right) = 0\end{aligned}$$

for $1 \le j \le m_{n-1}$. This again reflects the fact that the symmetry of the Dynkin diagram of type D_n is unbroken at level-0. One might expect that a nesting procedure of the type employed in [MR97] for rational closed spin chains and in [Go18] for rational open spin chains, in which the chain of symmetry subalgebras is $D_n \supset D_{n-1} \supset \cdots \supset D_1$, would preserve this symmetry of the Dynkin diagram at all levels of nesting.

Remark 4.2.11. For $n = 2$, the Bethe equations (4.2.38) and (4.2.40) decouple into two sets of Bethe equations for $U_q(\mathfrak{sl}_2)$-symmetric spin chains, and can be solved separately. This is consistent with the isomorphism $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. Similarly, for $n = 3$, the isomorphism $\mathfrak{so}_6 \cong \mathfrak{sl}_4$ is borne out in the Bethe equations (4.2.37), (4.2.38) and (4.2.40).

Remark 4.2.12. Let a_{ij} denote the matrix entries of a connected Dynkin diagram of type C_n or D_n and let I denote the set of its nodes. Then put $d_1 = \ldots = d_n = 1$ except $d_n = 2$ for C_n. Upon substituting $u_j^{(k)} \to q^{\widetilde{d}_k} z_j^{(k)}$, where $\widetilde{d}_k = \sum_{i=1}^{k} d_i$ except $\widetilde{d}_n = \sum_{i=1}^{n-1} d_i$ for D_n, and taking into account (4.1.35) and (4.1.36), Bethe equations above can be written as

$$\prod_{i=1}^{\ell} \frac{\lambda_k(q^{\widetilde{d}_k} z_j^{(k)})}{\lambda_{k+1}(q^{\widetilde{d}_k} z_j^{(k)})} = -\prod_{l\in I}\prod_{i=1}^{m_l} \frac{z_j^{(k)} - q^{d_k a_{kl}} z_i^{(l)}}{q^{d_k a_{kl}} z_j^{(k)} - z_i^{(l)}},$$

for $1 \le k \le n$ and all allowed j.

4.2.3 A nearest-neighbour interaction Hamiltonian

In the case where $L(\lambda^{(i)})_{c_i} \cong \mathbb{C}^n$, so that $L \cong (\mathbb{C}^n)^{\otimes \ell}$ and the Lax operators are given simply by the R-matrix (4.1.7), a nearest neighbour spin chain Hamiltonian may be extracted from the transfer matrix by taking the logarithmic derivative at a particular value of the spectral parameter v. Indeed, set $c_i = 1$ for $1 \le i \le \ell$ and define an adjusted transfer matrix by

$$t(v) := \left(\frac{1-v}{q-q^{-1}}\right)^{\ell} \tau(v),$$

with the property that at $v = 1$ it becomes the shift operator,

$$t(1) = \mathrm{tr}_a\, P_{a1} P_{a2} \cdots P_{a\ell} = P_{\ell-1,\ell} P_{\ell-2,\ell-1} \cdots P_{12}.$$

A nearest-neighbour interaction Hamiltonian is then

$$H := \frac{d}{dv} \ln t(v)\Big|_{v=1} = \big(t(1)^{-1}\big)\, t'(1) = \sum_{i=1}^{\ell-1} h_{i,i+1} + h_{\ell,1},$$

where the interaction between adjacent sites $h \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ is given by

$$h := I - \frac{P R_q}{q - q^{-1}} + \frac{P Q_q}{q^{2\kappa} - 1}.$$

4.2.4 Trace formula for Bethe vectors

Introduce matrices $f_\theta \in \mathrm{End}(\mathbb{C}^{2n})$ by $f_+ = q^{-1}e_{n+2,n} - q\,e_{n+1,n-1}$ and $f_- = -q\,e_{n+1,n}$. Then define a transposition ω on $\mathrm{End}(\mathbb{C}^{2n})$ by $\omega : e_{ij} \mapsto \theta_{ij} q^{\nu_i - \nu_j} e_{\bar{\jmath}\bar{\imath}}$ where $\bar{\imath} = 2n - i + 1$ and $\bar{\jmath} = 2n - j + 1$. The Theorem below is our second main result.

Theorem 4.2.13. *The level-0 Bethe vector can be written as*

$$\begin{aligned}\Phi_\theta^{(0)}(\boldsymbol{u}^{(0\dots n-1)}) = \mathrm{tr}_{\overline{W}} \Bigg[& \left(\prod_{k=1}^{n-1} \prod_{i=1}^{m_k} \prod_{j=1}^{m_0} R_{q^{-1},a_i^k a_j^0}(u_i^{(k)}, q^{2\theta} u_j^{(0)}) \right) \\ & \times \left(\prod_{i=1}^{m_0} T^\omega_{a_i^0}(u_i^{(0)}) \right) \left(\prod_{k=1}^{n-1} \prod_{i=1}^{m_k} \prod_{j=1}^{m_0} R_{a_i^k a_j^0}(u_i^{(k)}, q^{-2\kappa} u_j^{(0)}) \right) \\ & \times \left(\prod_{k=1}^{n-1} \prod_{i=1}^{m_k} T_{a_i^k}(u_i^{(k)}) \right) \left(\prod_{k=2}^{n-1} \prod_{l=1}^{k-1} \prod_{i=1}^{m_k} \prod_{j=m_l}^{1} R_{a_i^k a_j^l}(u_i^{(k)}, u_j^{(l)}) \right) \\ & \times (f_\theta)^{\otimes m_0} \otimes (e_{21})^{\otimes m_1} \otimes \cdots \otimes (e_{n,n-1})^{\otimes m_{n-1}} \Bigg] \cdot \eta, \end{aligned} \tag{4.2.41}$$

where the trace is taken over the space $\overline{W} = W_{\boldsymbol{a}^0} \otimes \cdots \otimes W_{\boldsymbol{a}^{n-1}} \cong (\mathbb{C}^{2n})^{\otimes(m_0+\ldots+m_{n-1})}$.

Proof. Recall the trace formula for the Bethe vectors of a $U_q(\mathfrak{gl}_n)$-symmetric spin chain given in Section 5.2 of [BR08]. This result implies the following formula for the level-1 Bethe vector:

$$\begin{aligned}\Phi_\theta^{(1)}(\boldsymbol{u}^{(1\dots n-1)}; \boldsymbol{u}^{(0)}) = \mathrm{tr}_{\overline{W}^{(1)}} \Bigg[& \left(\prod_{k=1}^{n-1} \prod_{i=1}^{m_k} A^{(1)}_{a_i^k \boldsymbol{a}^{\bar{0},0}}(u_i^{(k)}; \boldsymbol{u}^{(0)}) \right) \\ & \times \left(\prod_{k=2}^{n-1} \prod_{l=1}^{k-1} \prod_{i=1}^{m_k} \prod_{j=m_l}^{1} R^{(1,1)}_{a_i^k a_j^l}(u_i^{(k)}, u_j^{(l)}) \right) \\ & \times (e^{(1)}_{21})^{\otimes m_1} \otimes \cdots \otimes (e^{(1)}_{n,n-1})^{\otimes m_{n-1}} \Bigg] \cdot \eta^{(1)}_{-\theta}, \end{aligned}$$

where the trace is taken over the space $\overline{W}^{(1)} = W^{(1)}_{a_1^1} \otimes \cdots \otimes W^{(1)}_{a^{n-1}_{m_{n-1}}} \cong (\mathbb{C}^n)^{\otimes(m_1+\ldots+m_{n-1})}$. From

(4.2.4), this is equal to

$$\Phi^{(1)}_{\theta}(\boldsymbol{u}^{(1\ldots n-1)};\boldsymbol{u}^{(0)}) = \mathrm{tr}_{\overline{W}^{(1)}}\Bigg[\left(\prod_{k=1}^{n-1}\prod_{i=1}^{m_k}\prod_{j=m_0}^{1} K^{(1,1)}_{a^k_i\tilde{a}^0_j}(u^{(k)}_i, q^{2\theta}u^{(0)}_j)\right)$$
$$\times\left(\prod_{k=1}^{n-1}\prod_{i=1}^{m_k}\prod_{j=1}^{m_0} K^{(1,1)}_{a^k_i a^0_j}(u^{(k)}_i, u^{(0)}_j)\right)$$
$$\times\left(\prod_{k=1}^{n-1}\prod_{i=1}^{m_k} A^{(1)}_{a^k_i}(u^{(k)}_i)\right)\left(\prod_{k=2}^{n-1}\prod_{l=1}^{k-1}\prod_{i=1}^{m_k}\prod_{j=m_l}^{1} R^{(1,1)}_{a^k_i a^l_j}(u^{(k)}_i, u^{(l)}_j)\right)$$
$$\times (e^{(1)}_{21})^{\otimes m_1}\otimes\cdots\otimes(e^{(1)}_{n,n-1})^{\otimes m_{n-1}}\Bigg]\cdot\eta^{(1)}_{-\theta}.$$

We now introduce the level-0 creation operators in order to arrive at an expression for the level-0 Bethe vector, as given in Definition 4.2.6,

$$\Phi^{(0)}_{\theta}(\boldsymbol{u}^{(0\ldots n-1)}) = \mathrm{tr}_{\overline{W}^{(1)}}\Bigg[\left(\prod_{i=1}^{m_0}\beta_{\tilde{a}^0_i a^0_i}(u^{(0)}_i)\right)\left(\prod_{k=1}^{n-1}\prod_{i=1}^{m_k}\prod_{j=m_0}^{1} K^{(1,1)}_{a^k_i\tilde{a}^0_j}(u^{(k)}_i, q^{2\theta}u^{(0)}_j)\right)$$
$$\times\left(\prod_{k=1}^{n-1}\prod_{i=1}^{m_k}\prod_{j=1}^{m_0} K^{(1,1)}_{a^k_i a^0_j}(u^{(k)}_i, u^{(0)}_j)\right)\left(\prod_{k=1}^{n-1}\prod_{i=1}^{m_k} A^{(1)}_{a^k_i}(u^{(k)}_i)\right)$$
$$\times\left(\prod_{k=2}^{n-1}\prod_{l=1}^{k-1}\prod_{i=1}^{m_k}\prod_{j=m_l}^{1} R^{(1,1)}_{a^k_i a^l_j}(u^{(k)}_i, u^{(l)}_j)\right)$$
$$\times (e^{(1)}_{21})^{\otimes m_1}\otimes\cdots\otimes(e^{(1)}_{n,n-1})^{\otimes m_{n-1}}\Bigg]\cdot\eta^{(1)}_{-\theta}.$$

The next step is to rewrite the above expression in terms of the matrix $B(u)$, cf. (4.1.23), rather than the creation operator $\beta(u)$. Consider the following in expression, in which a matrix operator X acts non-trivially on the space $\tilde{a}^0$ and trivially on the space a^0, and vice versa for the matrix operator Y, and both operators act non-trivially on any number of other spaces,

$$\beta_{\tilde{a}^0 a^0}(u)\,X_{\tilde{a}^0}\,Y_{a^0}\cdot(e^{(1)}_k)_{\tilde{a}^0}\otimes(e^{(1)}_l)_{a^0} = \sum_{ij}(e^{(1)*}_j)_{\tilde{a}^0}\otimes(e^{(1)*}_i)_{a^0}\otimes q^{-i}b_{\bar{\imath}\bar{\jmath}}(u)\cdot X_{\tilde{a}^0}\,Y_{a^0}\cdot(e^{(1)}_k)_{\tilde{a}^0}\otimes(e^{(1)}_l)_{a^0}.$$

Contracting matrices gives

$$\begin{aligned}\sum_{ij} q^{-i}\,b_{\bar{\imath}\bar{\jmath}}(u)\,[X]_{jk}[Y]_{il} &= \sum_{ij} q^{-\bar{\jmath}}\big[B^{\omega}(u)\big]_{\bar{\jmath}i}[X]_{jk}[Y]_{il}\\ &= \sum_{ij} q^{-\bar{k}}\big[B^{\omega}(u)\big]_{\bar{\jmath}i}[X^{\omega}]_{\bar{k}\bar{\jmath}}[Y]_{il}\\ &= \sum_{ij} q^{-\bar{k}}[X^{\omega}]_{\bar{k}\bar{\jmath}}\big[B^{\omega}(u)\big]_{\bar{\jmath}i}[Y]_{il}\end{aligned}$$

$$= q^{-\bar{k}}[X^\omega B^\omega(u) Y]_{\bar{k}l} = q^{-\bar{k}} \operatorname{tr}\big[X^\omega B^\omega(u) Y e^{(1)}_{l\bar{k}}\big].$$

We have thus arrived at the identity

$$\beta_{\tilde{a}^0 a^0}(u) X_{\tilde{a}^0} Y_{a^0} \cdot (e^{(1)}_k)_{\tilde{a}^0} \otimes (e^{(1)}_l)_{a^0} = q^{-\bar{k}} \operatorname{tr}\big[X^\omega B^\omega(u) Y e^{(1)}_{l\bar{k}}\big].$$

Now recall (4.2.9). Hence we need to consider the cases when $(k,l) = (1,1),(1,2),(2,1)$, or equivalently $(l,\bar{k}) = (1,n),(2,n),(1,n-1)$. Bearing this in mind we define matrices $f^{(1)}_\theta \in \operatorname{End}(\mathbb{C}^n)$ by $f^{(1)}_- = -q\, e^{(1)}_{1,n}$ and $f^{(1)}_+ = q^{-1} e^{(1)}_{2,n} - q\, e^{(1)}_{1,n-1}$. This allows us to write the level-0 Bethe vector as follows:

$$\begin{aligned}
&\Phi^{(0)}_\theta(\boldsymbol{u}^{(0\ldots n-1)}) \\
&\quad = \operatorname{tr}_{\overline{W}}\Bigg[\Bigg(\prod_{k=1}^{n-1}\prod_{i=1}^{m_k}\prod_{j=m_0}^{1} K^{(1,1)}_{a^k_i a^0_j}(u^{(k)}_i, q^{2\theta}u^{(0)}_j)\Bigg)^{\omega_{\boldsymbol{a}^0}} \\
&\qquad\qquad \times \Bigg(\prod_{i=1}^{m_0} B^\omega_{a^0_i}(u^{(0)}_i)\Bigg)\Bigg(\prod_{k=1}^{n-1}\prod_{i=1}^{m_k}\prod_{j=1}^{m_0} K^{(1,1)}_{a^k_i a^0_j}(u^{(k)}_i, u^{(0)}_j)\Bigg) \\
&\qquad\qquad \times \Bigg(\prod_{k=1}^{n-1}\prod_{i=1}^{m_k} A^{(1)}_{a^k_i}(u^{(k)}_i)\Bigg)\Bigg(\prod_{k=2}^{n-1}\prod_{l=1}^{k-1}\prod_{i=1}^{m_k}\prod_{j=m_l}^{1} R^{(1,1)}_{a^k_i a^l_j}(u^{(k)}_i, u^{(l)}_j)\Bigg) \\
&\qquad\qquad \times (\theta q^{\theta-n} f^{(1)}_\theta)^{\otimes m_0} \otimes (e^{(1)}_{21})^{\otimes m_1} \otimes \cdots \otimes (e^{(1)}_{n,n-1})^{\otimes m_{n-1}}\Bigg] \cdot \eta \\
&\quad = \operatorname{tr}_{\overline{W}}\Bigg[\Bigg(\prod_{k=1}^{n-1}\prod_{i=1}^{m_k}\prod_{j=1}^{m_0} R^{(1,1)}_{q^{-1},a^k_i a^0_j}(u^{(k)}_i, q^{2\theta}u^{(0)}_j)\Bigg) \\
&\qquad\qquad \times \Bigg(\prod_{i=1}^{m_0} \theta q^{\theta-n} B^\omega_{a^0_i}(u^{(0)}_i)\Bigg)\Bigg(\prod_{k=1}^{n-1}\prod_{i=1}^{m_k}\prod_{j=1}^{m_0} K^{(1,1)}_{a^k_i a^0_j}(u^{(k)}_i, u^{(0)}_j)\Bigg) \\
&\qquad\qquad \times \Bigg(\prod_{k=1}^{n-1}\prod_{i=1}^{m_k} A^{(1)}_{a^k_i}(u^{(k)}_i)\Bigg)\Bigg(\prod_{k=2}^{n-1}\prod_{l=1}^{k-1}\prod_{i=1}^{m_k}\prod_{j=m_l}^{1} R^{(1,1)}_{a^k_i a^l_j}(u^{(k)}_i, u^{(l)}_j)\Bigg) \\
&\qquad\qquad \times (f^{(1)}_\theta)^{\otimes m_0} \otimes (e^{(1)}_{21})^{\otimes m_1} \otimes \cdots \otimes (e^{(1)}_{n,n-1})^{\otimes m_{n-1}}\Bigg] \cdot \eta.
\end{aligned} \tag{4.2.42}$$

It remains to show that the form of the Bethe vector given in (4.2.41) reduces to to above form, by considering the decomposition $\mathbb{C}^{2n} \cong \mathbb{C}^2 \otimes \mathbb{C}^n$ as in (4.1.12), and tracing out all the $\mathbb{C}^2$ spaces.

Making explicit this decomposition, the formula (4.2.41) becomes

$$
\begin{aligned}
\Phi_\theta^{(0)}(\boldsymbol{u}^{(0\dots n-1)}) = \mathrm{tr}_{\overline{W}} \Bigg[& \left(\prod_{k=1}^{n-1} \prod_{i=1}^{m_k} \prod_{j=1}^{m_0} R_{q^{-1},a_i^k a_j^0}(u_i^{(k)}, q^{2\theta} u_j^{(0)}) \right) \\
& \times \left(\prod_{i=1}^{m_0} T^\omega_{a_i^0}(u_i^{(0)}) \right) \left(\prod_{k=1}^{n-1} \prod_{i=1}^{m_k} \prod_{j=1}^{m_0} R_{a_i^k a_j^0}(u_i^{(k)}, q^{-2\kappa} u_j^{(0)}) \right) \\
& \times \left(\prod_{k=1}^{n-1} \prod_{i=1}^{m_k} T_{a_i^k}(u_i^{(k)}) \right) \left(\prod_{k=2}^{n-1} \prod_{l=1}^{k-1} \prod_{i=1}^{m_k} \prod_{j=m_l}^{1} R_{a_i^k a_j^l}(u_i^{(k)}, u_j^{(l)}) \right) \\
& \times (x_{21} \otimes f_\theta^{(1)})^{\otimes m_0} \otimes (x_{11} \otimes e_{21}^{(1)})^{\otimes m_1} \otimes \dots \otimes (x_{11} \otimes e_{n,n-1}^{(1)})^{\otimes m_{n-1}} \Bigg] \cdot \eta.
\end{aligned}
\tag{4.2.43}
$$

Recall (4.1.13) and note that

$$
\begin{aligned}
R(u,v)\; x_{11} \otimes x_{11} &= R^{(1,1)}(u,v) x_{11} \otimes x_{11}, \\
R(u,v)\; x_{11} \otimes x_{21} &= U^{(1,1)}(u,v) x_{21} \otimes x_{11} + K^{(1,1)}(u, q^{2\kappa} v) x_{11} \otimes x_{21}.
\end{aligned}
$$

Next, recall (4.1.3) and note that

$$
\nu_{n+j} - \nu_i = j - \theta' - (-n + i - 1 + \theta') = j - i + n - \theta
$$

for all $1 \le i, j \le n$. This implies the following relationship between the transposition ω on $\mathrm{End}(\mathbb{C}^{2n})$ and its counterpart on $\mathrm{End}(\mathbb{C}^n)$:

$$
\big[T^\omega(u)\big]_{i,n+j} = \theta q^{i-j-n+\theta} t_{n-j+1,2n-i+1}(u) = \theta q^{\theta-n} \big[B^\omega(u)\big]_{ij}.
$$

Hence, the action of $T(u)$ and $T^\omega(u)$ on x_{11} and x_{21} takes the form

$$
\begin{aligned}
T(u) x_{11} &= A(u) x_{11} + C(u) x_{21}, \\
T^\omega(u) x_{21} &= \theta q^{\theta-n} B^\omega(u) x_{11} + A^\omega(u) x_{21}.
\end{aligned}
$$

The identities above imply that the numbers of x_{11}'s and x_{21}'s inside the trace in (4.2.43) are conserved individually under the action R-matrices. Therefore, the only possibility for the partial trace over the $\mathbb{C}^2$ spaces to be nonzero is if the action of the $T^\omega_{a_i^0}(u_i^{(0)})$ maps $(x_{21})_{a_i^0}$ to $(x_{11})_{a_i^0}$. That is, each $T^\omega_{a_i^0}(u_i^{(0)})$ acts as $q^{\theta-n}\theta B^\omega_{a_i^0}(u_i^{(0)})$. Since each $(x_{21})_{a_i^0}$ must be acted on by $T^\omega_{a_i^0}(u_i^{(0)})$, the R-matrices to the right of the $T^\omega_{a_i^0}(u_i^{(0)})$'s must not permute the $(x_{21})_{a_j^0}$ with the $(x_{11})_{a_i^k}$ for $k \ge 1$. That is, the $R_{a_i^k a_j^0}(u_i^{(k)}, q^{-2\kappa} u_j^{(0)})$ each act as $K^{(1,1)}_{a_i^k a_j^0}(u_i^{(k)}, u_j^{(0)})$ in order for the trace to be non-zero. Finally, all other R-matrices in (4.2.43) act on suitable pairs $x_{11} \otimes x_{11}$ only, and may simply be replaced with $R^{(1,1)}(u,v)$, for appropriate u, v. This proves that taking the partial trace over the $\mathbb{C}^2$ spaces in (4.2.43) we arrive at (4.2.42), as required. $\square$

Below we provide two examples of level-0 Bethe vectors obtained using (4.2.43). We will assume that $m_i = 0$ for $0 \le i \le n-1$ if not stated otherwise. We also use the notation $p_k := \delta_{k,n-1}\, q^{-2}\,(q - q^{-1})$.

Example 4.2.14 (Symplectic case). For $n \ge 1$ and $m_0 = m \ge 1$ we have

$$\Phi_-^{(0)}(u_1^{(0)}, \dots, u_m^{(0)}) = q^{-m} t_{n,n+1}(u_1^{(0)}) \cdots t_{n,n+1}(u_m^{(0)}) \cdot \eta.$$

For $n \ge 2$ and $m_0 = m_k = 1$ with $1 \le k \le n-1$ we have

$$\Phi_-^{(0)}(u_1^{(0)}, u_1^{(k)}) = q^{-1} \Bigg[t_{n,n+1}(u_1^{(0)})\, t_{k,k+1}(u_1^{(k)}) \\ + \frac{p_k\, u_1^{(0)}}{u_1^{(k)} - u_1^{(0)}} \Big(t_{n-1,n+1}(u_1^{(0)}) + t_{n,n+2}(u_1^{(0)}) \Big)\, t_{nn}(u_1^{(k)}) \Bigg] \cdot \eta.$$

For $n \ge 2$ and $m_0 = 2$, $m_k = 1$ with $1 \le k \le n-1$ we have

$$\Phi_-^{(0)}(u_1^{(0)}, u_2^{(0)}, u_1^{(k)}) = q^{-2} \Bigg[t_{n,n+1}(u_1^{(0)})\, t_{n,n+1}(u_2^{(0)})\, t_{k,k+1}(u_1^{(k)}) \\ + p_k \Bigg(\frac{u_2^{(0)}}{u_1^{(k)} - u_2^{(0)}}\, t_{n,n+1}(u_1^{(0)}) \Bigg(t_{n-1,n+1}(u_2^{(0)}) + \frac{q u_1^{(k)} - q^{-1} u_1^{(0)}}{u_1^{(k)} - u_1^{(0)}}\, t_{n,n+2}(u_2^{(0)}) \Bigg) \\ + \frac{u_1^{(0)}}{u_1^{(k)} - u_1^{(0)}} \Bigg(\frac{q u_1^{(k)} - q^{-1} u_2^{(0)}}{u_1^{(k)} - u_2^{(0)}}\, t_{n-1,n+1}(u_1^{(0)}) + \frac{q^2 u_1^{(k)} - q^{-2} u_2^{(0)}}{u_1^{(k)} - u_2^{(0)}}\, t_{n,n+2}(u_1^{(0)}) \Bigg) \\ \times t_{n,n+1}(u_2^{(0)}) \Bigg) t_{k+1,k+1}(u_1^{(k)}) \Bigg] \cdot \eta.$$

Example 4.2.15 (Orthogonal case). For $n \ge 1$ and $m_0 = m \ge 1$ we have

$$\Phi_+^{(0)}(u_1^{(0)}, \dots, u_m^{(0)}) = \prod_{k=1}^{m} \Big(q^{-2} t_{n-1,n+1}(u_k^{(0)}) - t_{n,n+2}(u_k^{(0)}) \Big) \cdot \eta.$$

For $n \ge 2$ and $m_0 = m_k = 1$ with $1 \le k \le n-1$ we have

$$\Phi_+^{(0)}(u_1^{(0)}, u_1^{(k)}) = \Bigg[\frac{q^{\delta_{k,n-1}} u_1^{(k)} - q^{-\delta_{k,n-1}} u_1^{(0)}}{q^2 (u_1^{(k)} - u_1^{(0)})} \Big(q^{-2} t_{n-1,n+1}(u_1^{(0)}) - t_{n,n+2}(u_1^{(0)}) \Big)\, t_{k,k+1}(u_1^{(k)}) \\ + \frac{p_{k+1}\, u_1^{(0)}}{u_1^{(k)} - u_1^{(0)}} \Big(q^{-2} t_{n-2,n+1}(u_1^{(0)}) - q\, t_{n,n+3}(u_1^{(0)}) \Big)\, t_{k+1,k+1}(u_1^{(k)}) \Bigg] \cdot \eta.$$

For $n \geq 4$ and $m_0 = 2$, $m_k = 1$ with $1 \leq k \leq n-3$ we have

$$\Phi_+^{(0)}(u_1^{(0)}, u_2^{(0)}, u_1^{(k)}) = \left(q^{-2} t_{n-1,n+1}(u_1^{(0)}) - t_{n,n+2}(u_1^{(0)})\right) \\ \times \left(q^{-2} t_{n-1,n+1}(u_2^{(0)}) - t_{n,n+2}(u_2^{(0)})\right) t_{k,k+1}(u_1^{(k)}) \cdot \eta.$$

The $k = n-2$ and $k = n-1$ cases have long tails, hence we have not written them out explicitly.

4.2.5 The semi-classical limit

In order to retrieve the results of [Rs85] and [DVK87], we investigate the semi-classical $q \to 1$ limit, or equivalently the $\hbar \to 0$ limit. The limit must be taken in a particular way, as the spectral parameters have a hidden q dependence. Setting $u = e^{2x\hbar}$, $v = e^{2y\hbar}$ and $q = e^{\hbar}$ and expanding in powers of $\hbar$, we recover the Zamolodchikov R-matrix [ZZ78, KS82],

$$R_q \underset{\hbar \to 0}{\longrightarrow} I, \qquad Q_q \underset{\hbar \to 0}{\longrightarrow} Q := \sum_{i,j=1}^{2n} e_{ij} \otimes e_{\bar{i}\bar{j}}, \qquad R(u,v) \underset{\hbar \to 0}{\longrightarrow} R(x-y) := I - \frac{P}{x-y} - \frac{Q}{\kappa - (x-y)}.$$

The reduced R-matrices become the Yang R-matrices,

$$R^{(k,l)}(u,v) \underset{\hbar \to 0}{\longrightarrow} R^{(k,l)}(x-y) := I^{(k,l)} - \frac{P^{(k,l)}}{x-y}.$$

The eigenvalues given in Theorem 4.2.8 in the limit become

$$\begin{aligned}
\Lambda^{(1)}(y; \boldsymbol{x}^{(1\ldots n-1)}; \boldsymbol{x}^{(0)}) &= \prod_{i=1}^{m_1} \frac{y - x_i^{(1)} + 1}{y - x_i^{(1)}} \prod_{i=1}^{\ell} \lambda_1^{(i)}(y) \\
&+ \sum_{k=2}^{n-2} \prod_{i=1}^{m_{k-1}} \frac{y - x_i^{(k-1)} - 1}{y - x_i^{(k-1)}} \prod_{i=1}^{m_k} \frac{y - x_i^{(k)} + 1}{y - x_i^{(k)}} \prod_{i=1}^{\ell} \lambda_k^{(i)}(y) \\
&+ \prod_{i=1}^{m_{n-2}} \frac{y - x_i^{(n-2)} - 1}{y - x_i^{(n-2)}} \prod_{i=1}^{m_{n-1}} \frac{y - x_i^{(n-1)} + 1}{y - x_i^{(n-1)}} \prod_{i=1}^{m_0} \frac{y - x_i^{(0)} + \theta'}{y - x_i^{(0)}} \prod_{i=1}^{\ell} \lambda_{n-1}^{(i)}(y) \\
&+ \prod_{i=1}^{m_{n-1}} \frac{y - x_i^{(n-1)} - 1}{y - x_i^{(n-1)}} \prod_{i=1}^{m_0} \frac{y - x_i^{(0)} + 2 - \theta'}{y - x_i^{(0)}} \prod_{i=1}^{\ell} \lambda_n^{(i)}(y)
\end{aligned}$$

and

$$
\begin{aligned}
\widetilde{\Lambda}^{(1)}(y;\boldsymbol{x}^{(1\ldots n-1)};\boldsymbol{x}^{(0)}) &= \prod_{i=1}^{m_1}\frac{y-x_i^{(1)}-\kappa}{y-x_i^{(1)}-\kappa+1}\prod_{i=1}^{\ell}\lambda_{2n}^{(i)}(y) \\
&+\sum_{k=2}^{n-2}\prod_{i=1}^{m_{k-1}}\frac{y-x_i^{(k-1)}-\kappa+k}{y-x_i^{(k-1)}-\kappa+k-1}\prod_{i=1}^{m_k}\frac{y-x_i^{(k)}-\kappa+k-1}{y-x_i^{(k)}-\kappa+k}\prod_{i=1}^{\ell}\lambda_{2n-k+1}^{(i)}(y) \\
&+\prod_{i=1}^{m_{n-2}}\frac{y-x_i^{(n-2)}+\theta-1}{y-x_i^{(n-2)}+\theta-2}\prod_{i=1}^{m_{n-1}}\frac{y-x_i^{(n-1)}+\theta-1}{y-x_i^{(n-1)}+\theta-2}\prod_{i=1}^{m_0}\frac{y-x_i^{(0)}-\theta'}{y-x_i^{(0)}}\prod_{i=1}^{\ell}\lambda_{n+2}^{(i)}(y) \\
&+\prod_{i=1}^{m_{n-1}}\frac{y-x_i^{(n-1)}+\theta}{y-x_i^{(n-1)}+\theta-1}\prod_{i=1}^{m_0}\frac{y-x_i^{(0)}-2+\theta'}{y-x_i^{(0)}}\prod_{i=1}^{\ell}\lambda_{n+1}^{(i)}(y)
\end{aligned}
$$

where the rational weights are given by

$$
\lambda_j^{(i)}(v)\underset{\hbar\to 0}{\longrightarrow}\lambda_j^{(i)}(y):=\begin{cases}\dfrac{y-b_i-s_i}{y-b_i} & \text{if}\quad j=1,\\[2mm] 1 & \text{if}\quad 1<j<2n,\\[2mm] \dfrac{y-b_i-\kappa+1}{y-b_i-\kappa+1-s_i} & \text{if}\quad j=2n\end{cases}
\tag{4.2.44}
$$

in the symmetric case, i.e. when $\mathfrak{g}_{2n}=\mathfrak{so}_{2n}$, and by

$$
\lambda_j^{(i)}(v)\underset{\hbar\to 0}{\longrightarrow}\lambda_j^{(i)}(y):=\begin{cases}\dfrac{y-b_i-1}{y-b_i} & \text{if}\quad 1\le j\le s_i,\\[2mm] 1 & \text{if}\quad s_i<j<2n-s_i+1,\\[2mm] \dfrac{y-b_i-\kappa+s_i}{y-b_i-\kappa+s_i-1} & \text{if}\quad 2n-s_i+1\le j\le 2n\end{cases}
\tag{4.2.45}
$$

in the skewsymmetric case, i.e. when $\mathfrak{g}_{2n}=\mathfrak{sp}_{2n}$; here $b_i=\frac{1}{2\hbar}\log c_i\in\mathbb{C}$ are the inhomogeneities.

The Bethe equations may be obtained in the same way. Denoting $x_j^{(n)}:=x_j^{(0)}$ and $m_n:=m_0$ their explicit form is, in the symplectic case,

$$
\begin{aligned}
\prod_{i=1}^{\ell}\frac{\lambda_1^{(i)}(x_j^{(1)})}{\lambda_2^{(i)}(x_j^{(1)})} &= \prod_{\substack{i=1\\ i\neq j}}^{m_1}\frac{x_j^{(1)}-x_i^{(1)}-1}{x_j^{(1)}-x_i^{(1)}+1}\prod_{i=1}^{m_2}\frac{x_j^{(1)}-x_i^{(2)}+1}{x_j^{(1)}-x_i^{(2)}}, \\
\prod_{i=1}^{\ell}\frac{\lambda_k^{(i)}(x_j^{(k)})}{\lambda_{k+1}^{(i)}(x_j^{(k)})} &= \prod_{i=1}^{m_{k-1}}\frac{x_j^{(k)}-x_i^{(k-1)}}{x_j^{(k)}-x_i^{(k-1)}-1}\prod_{\substack{i=1\\ i\neq j}}^{m_k}\frac{x_j^{(k)}-x_i^{(k)}-1}{x_j^{(k)}-x_i^{(k)}+1}\prod_{i=1}^{m_{k+1}}\frac{x_j^{(k)}-x_i^{(k+1)}+1}{x_j^{(k)}-x_i^{(k+1)}}, \\
\prod_{i=1}^{\ell}\frac{\lambda_{n-1}^{(i)}(x_j^{(n-1)})}{\lambda_n^{(i)}(x_j^{(n-1)})} &= \prod_{i=1}^{m_{n-2}}\frac{x_j^{(n-1)}-x_i^{(n-2)}}{x_j^{(n-1)}-x_i^{(n-2)}-1}\prod_{\substack{i=1\\ i\neq j}}^{m_{n-1}}\frac{x_j^{(n-1)}-x_i^{(n-1)}-1}{x_j^{(n-1)}-x_i^{(n-1)}+1}\prod_{i=1}^{m_n}\frac{x_j^{(n-1)}-x_i^{(n)}+2}{x_j^{(n-1)}-x_i^{(n)}},
\end{aligned}
$$

$$\prod_{i=1}^{\ell}\frac{\lambda_n^{(i)}(x_j^{(n)})}{\lambda_{n+1}^{(i)}(x_j^{(n)})}=\prod_{i=1}^{m_{n-1}}\frac{x_j^{(n)}-x_i^{(n-1)}}{x_j^{(n)}-x_i^{(n-1)}-2}\prod_{\substack{i=1\\i\neq j}}^{m_n}\frac{x_j^{(n)}-x_i^{(n)}-2}{x_j^{(n)}-x_i^{(n)}+2}$$

for $2\leq k\leq n-2$ and all allowed j, and weights given by (4.2.45). In the orthogonal case, the Bethe equations for $k=1,\ldots,n-3$ are identical to the symplectic case; for $k=n-2,n-1,n$ they are, respectively,

$$\begin{aligned}\prod_{i=1}^{\ell}\frac{\lambda_{n-2}^{(i)}(x_j^{(n-2)})}{\lambda_{n-1}^{(i)}(x_j^{(n-2)})}&=\prod_{i=1}^{m_{n-3}}\frac{x_j^{(n-2)}-x_i^{(n-3)}}{x_j^{(n-2)}-x_i^{(n-3)}-1}\prod_{\substack{i=1\\i\neq j}}^{m_{n-2}}\frac{x_j^{(n-2)}-x_i^{(n-2)}-1}{x_j^{(n-2)}-x_i^{(n-2)}+1}\\&\quad\times\prod_{i=1}^{m_{n-1}}\frac{x_j^{(n-2)}-x_i^{(n-1)}+1}{x_j^{(n-2)}-x_i^{(n-1)}}\prod_{i=1}^{m_n}\frac{x_j^{(n-2)}-x_i^{(n)}+1}{x_j^{(n-2)}-x_i^{(n)}},\\\prod_{i=1}^{\ell}\frac{\lambda_{n-1}^{(i)}(x_j^{(n-1)})}{\lambda_{n}^{(i)}(x_j^{(n-1)})}&=\prod_{i=1}^{m_{n-2}}\frac{x_j^{(n-1)}-x_i^{(n-2)}}{x_j^{(n-1)}-x_i^{(n-2)}-1}\prod_{\substack{i=1\\i\neq j}}^{m_{n-1}}\frac{x_j^{(n-1)}-x_i^{(n-1)}-1}{x_j^{(n-1)}-x_i^{(n-1)}+1},\\\prod_{i=1}^{\ell}\frac{\lambda_{n-1}^{(i)}(x_j^{(n)})}{\lambda_{n+1}^{(i)}(x_j^{(n)})}&=\prod_{i=1}^{m_{n-2}}\frac{x_j^{(n)}-x_i^{(n-2)}}{x_j^{(n)}-x_i^{(n-2)}-1}\prod_{\substack{i=1\\i\neq j}}^{m_n}\frac{x_j^{(n)}-x_i^{(n)}-1}{x_j^{(n)}-x_i^{(n)}+1}\end{aligned}$$

for all allowed j, and weights given by (4.2.44). Substituting $x_j^{(k)}\to w_j^{(k)}-\widetilde{d}_k$ with $\widetilde{d}_k$ and assuming restrictions on n as in Remark 4.2.12, the Bethe equations for both symplectic and orthogonal cases take the form

$$\prod_{i=1}^{\ell}\frac{\lambda_k^{(i)}(w_j^{(k)}-\widetilde{d}_k)}{\lambda_{k+1}^{(i)}(w_j^{(k)}-\widetilde{d}_k)}=-\prod_{l\in I}\prod_{i\in 1}^{m_l}\frac{w_j^{(k)}-w_i^{(l)}-\frac{1}{2}d_ka_{kl}}{w_j^{(k)}-w_i^{(l)}+\frac{1}{2}d_ka_{kl}}$$

for $1\leq k\leq n$ and all allowed j.

Finally, the trace formula for level-0 Bethe vector (4.2.43) takes the form

$$\begin{aligned}\Phi^{(0)}(\boldsymbol{x}^{(0\ldots n-1)})=\mathrm{tr}_{\overline{W}}\Bigg[&\left(\prod_{k=1}^{n-1}\prod_{i=1}^{m_k}\prod_{j=1}^{m_0}R_{a_i^ka_j^0}(x_i^{(k)}-x_j^{(0)}-\theta)\right)\\&\times\left(\prod_{i=1}^{m_0}T^t_{a_i^0}(x_i^{(0)})\right)\left(\prod_{k=1}^{n-1}\prod_{i=1}^{m_k}\prod_{j=1}^{m_0}R_{a_i^ka_j^0}(x_i^{(k)}-x_j^{(0)}+\kappa)\right)\\&\times\left(\prod_{k=1}^{n-1}\prod_{i=1}^{m_k}T_{a_i^k}(x_i^{(k)})\right)\left(\prod_{k=2}^{n-1}\prod_{l=1}^{k-1}\prod_{i=1}^{m_k}\prod_{j=m_l}^{1}R_{a_i^ka_j^l}(x_i^{(k)}-x_j^{(l)})\right)\\&\times(f_\theta)^{\otimes m_0}\otimes(e_{21})^{\otimes m_1}\otimes\cdots\otimes(e_{n,n-1})^{\otimes m_{n-1}}\Bigg]\cdot\eta,\end{aligned}$$

where the trace is taken over the space $\overline{W}=W_{a_1^0}\otimes\cdots\otimes W_{a_{m_{n-1}}^{n-1}}\cong(\mathbb{C}^{2n})^{\otimes(m_0+\ldots+m_{n-1})}$ and $f_\theta\in\mathrm{End}(\mathbb{C}^{2n})$ is defined for orthogonal and symplectic cases respectively by $f_1=e_{n+1,n-1}-e_{n+2,n}$

and $f_{-1} = e_{n+1,n}$, and $T_a(x)$'s are defined via the rational fusion procedure analogous to that in Section 4.1.3 (see [IMO12] and Section 3.1 in [GR20a]) and t is the transposition defined by $t : e_{ij} \mapsto e_{\bar{j}\bar{i}}$.

Chapter 5

Conclusion and Outlook

In this thesis we have studied closed and open spin chains using the nested algebraic Bethe ansatz technique. More specifically, we have studied even orthogonal and symplectic ($\mathfrak{g}_{2n}$, or indeed $U_q(\mathfrak{g}_{2n})$) spin chains, and applied a nesting procedure which reduced the diagonalisation problem to well-known $\mathfrak{gl}_n$-type problem.

Having found expressions for the eigenstates, physical properties of the spin chain can be explored by calculating expressions for scalar products between eigenstates and other important states, as well as matrix elements known as form factors, in terms of the undetermined Bethe roots. For the closed Heisenberg spin chain, it was shown that overlaps reduce elegantly to a determinant form [Sl89], and this has recently been extended to closed spin chains with $\mathfrak{gl}_n$ symmetry [GLR20]. It is hoped that a similarly elegant form exists for the open spin chains.

The Bethe ansatz has a well known application in models relating to the AdS/CFT correspondence: Hamiltonians for spin chains of certain symmetry algebras, including $\mathfrak{so}(6)$, show up in 1-loop corrections to correlation functions in the conformal field theory. A recent paper [LGKLP20] details these calculations for certain open spin chains, including models of similar type to those studied in Chapter 3.

Here we outline some alternatives and extensions to the methods we have presented.

5.1 An alternative nesting procedure

When applying the algebraic Bethe ansatz, the main freedom we have is the ‘creation operator’ for the system—both its structure and the ordering of multiple creation operators. In the nested algebraic Bethe ansatz, the creation operator is chosen to facilitate the reduction of the system to a lower-rank case using the algebraic relations between the monodromy matrix elements. In this work in particular, we have focussed on the nesting procedure $\mathfrak{g}_{2n} \supset \mathfrak{gl}_n \supset \cdots \supset \mathfrak{gl}_2$. However, it is possible to work ‘from the other end of the Dynkin diagram’, employing instead the procedure $\mathfrak{g}_{2n} \supset \mathfrak{g}_{2n-2} \supset \cdots \supset \mathfrak{g}_2$. An advantage of this nesting procedure is that it allows one to study the odd orthogonal (type B) case in the same manner: $\mathfrak{so}_{2n+1} \supset \mathfrak{so}_{2n-1} \supset \cdots \supset \mathfrak{so}_3$, where a variant of the $\mathfrak{gl}_2$ algebraic Bethe ansatz is applied at the final step. This is a well-known technique, and has been applied to closed [MR97] and open spin chains [GKR05, Go18].

Another approach to the type B case, first introduced by Reshetikhin [Rs85], defines a monodromy matrix with the spinor representation as its auxiliary space. The resulting matrix is 2^N dimensional, and at each step cutting the matrix into equal sized block matrices corresponds to the nesting procedure above.

The [MR97] paper also contains the algebraic Bethe ansatz for the more general orthosymplectic spin chain. For open orthosymplectic spin chains, the Bethe equations were given in [AACDFR04], but the construction of eigenvectors remains an open problem.

5.2 Baxter's Q operator

One technique that has seen much success in studying solvable models was introduced by Baxter [Ba72]. In his landmark solution of the 8-vertex model Baxter does not construct eigenvectors for the transfer matrix. Rather, he defines an operator, denoted Q, which commutes with the transfer matrix and the action of which on the Bethe eigenvector is simply given by a polynomial with roots at each of the Bethe roots. This implies that the expression for the eigenvalue of the transfer matrix may be replaced by an operatorial relation between the transfer matrix and the Q operator: Baxter's T-Q relation. The Bethe equations then appear as an analyticity condition for the transfer matrix eigenvalues. This technique was adapted by Reshetikhin into the analytical Bethe ansatz [Rs83], which allows one to obtain the eigenvalues and Bethe equations without the use of the 'heavy machinery' of the coordinate or algebraic Bethe ansätze, and which saw much success in the study of spin chains [KS95, AACDFR03, ACDFR06].

An explicit construction of Baxter's Q operator is not necessary for finding the transfer matrix eigenvalues and so, until recently, its form has only been known in the low-rank cases. However, in [BFLMS11] the $\mathfrak{gl}_n$ closed chain was studied in detail, linking the Q operator to infinite dimensional oscillator representations of (slight generalisations of) $Y(\mathfrak{gl}_n)$. It was shown that the Q operator may be constructed in the same way as the transfer matrix, but with the 'auxiliary space' being this oscillator representation. The investigation into even orthogonal spin chains is far advanced [Fr20], and it is hoped that these results could be generalised to open chains—the XXX and XXZ cases were studied in [FS15, VW20]—in particular to those with non-diagonal boundary conditions.

5.3 Solutions to the Bethe equations

In this work, our starting point has been a spin chain and transfer matrix, and our end point has been Bethe vectors, eigenvalues and Bethe equations. However, we did not touch on the notion of completeness of the Bethe ansatz, or the solutions to the Bethe equations themselves. Indeed, the counting of solutions to the Bethe equations has always been a topic of study, since even the original Bethe ansatz [Be31]; perhaps Bloch's [Bl30] miscounting of eigenvectors partially inspired Bethe to pick up the problem in the first place.

The Bethe ansatz completeness problem has been studied in detail in a series of works by Tarasov, Varchenko and Mukhin; [Ta18] contains the results for the XXX model as well as the relevant references. Higher rank cases have been studied using the 'reproduction procedure', a method which gives families of solutions originating from a single solution [MV03]. It would be interesting to explore the reproduction procedure for Bethe equations derived from open spin chains and determine of the mathematical structure of the corresponding population of solutions.

The counting and classifying of solutions into those that lead to physically valid eigenstates has also been studied using computational algebraic geometry [JZ18]. Similarly, in [MV17] a computational method was given for determining the Bethe roots using relations between the eigenvalues of the Baxter Q operator, known as the QQ system.

Regarding the models we have studied in this thesis, one natural prerequisite for the completeness of the Bethe ansatz is the irreducibility of the spin chain as a representation of the quantum group. In the case of the Yangian $Y(\mathfrak{gl}_n)$, this supplies a restriction on the shift parameters c_j and weights λ of the evaluation modules that make up the spin chain, mentioned briefly in Section 1.2.1. However, the relevant irreducibility conditions for representations of twisted Yangians are not yet known.

5.4 Separation of variables

Separation of variables is a well-known technique when applied to PDEs, but Sklyanin was the first to popularise it as a universal method for solving classical and quantum integrable models, including quantum spin chains [Sk92]. The method involves a detailed analysis of the B operator (creation operator), regarding it as a polynomial with operator coefficients and finding its operator roots. Diagonalising these operator roots, which commute by virtue of the algebra relations, then provides a useful basis for the construction of transfer matrix eigenvectors.

Sklyanin was able to apply the method to the $\mathfrak{su}_2$- and $\mathfrak{su}_3$-symmetric closed spin chains. For the $\mathfrak{su}_3$ model, the technique differs from the nested algebraic Bethe ansatz—rather than using a particular ordering of creation operators, the eigenstates are built from a single, carefully constructed B operator. The form of this operator was deduced from its classical counterpart, but this approach is limited by operator ordering ambiguities when moving to the quantum case. Nevertheless, the extension of this construction to $\mathfrak{su}_n$ was recently discovered [GLS17]. The orthogonal and symplectic cases remain open problems, as it is a difficult task to find an appropriate diagonalisable B operator with the correct number of roots.

Separation of variables has also been expanded to open spin chains [FKN14], where it was successfully applied to an XXZ system with arbitrary non-diagonal boundaries. We hope that in the future this technique can be applied to open spin chains of higher rank, as well as the orthogonal and symplectic spin chains.

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